# DIRECTORATE OF DISTANCE EDUCATION

# UNIVERSITY OF NORTH BENGAL

## MASTER OF SCIENCES- MATHEMATICS

## SEMESTER -III

## FUNCTIONAL ANALYSIS

## DEMATH3CORE2

# BLOCK-2

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# FOREWORD

The Self Learning Material (SLM) is written with the aim of providing simple and organized study content to all the learners. The SLMs are prepared on the framework of being mutually cohesive, internally consistent and structured as per the university's syllabi. It is a humble attempt to give glimpses of the various approaches and dimensions to the topic of study and to kindle the learner's interest to the subject

We have tried to put together information from various sources into this book that has been written in an engaging style with interesting and relevant examples. It introduces you to the insights of subject concepts and theories and presents them in a way that is easy to understand and comprehend.

We always believe in continuous improvement and would periodically update the content in the very interest of the learners. It may be added that despite enormous efforts and coordination, there is every possibility for some omission or inadequacy in few areas or topics, which would definitely be rectified in future.

We hope you enjoy learning from this book and the experience truly enrich your learning and help you to advance in your career and future endeavours.

# FUNCTIONAL ANALYSIS

## BLOCK -1



## BLOCK-2

# BLOCK-2 FUNCTIONAL ANALYSIS

**Functional analysis** was born in the early years of the twentieth century as part of a larger trend toward abstraction—what some authors have called the “arithmetization” of **analysis**. This same trend toward “axiomatics” contributed to the foundations of abstract linear algebra, modern geometry, and topology.

Finally we come to nonlinear functional analysis. In the past seven chapters we have been concerned with the properties of infinite dimensional spaces and linear operators including linear functionals between them. In some sense we were working on generalizations of linear algebra. In analysis one studies linear and nonlinear functions as well. Of course, there are far more nonlinear functions than linear functions. You just have to recall only degree one polynomials are linear and all polynomials of higher degree are nonlinear, let alone transcendental functions. In this chapter we give a very brief introduction to nonlinear functional analysis. The main theme is to extend results in calculus, especially in differentiation theory, to infinite dimensional settings. You will see that linearization plays a dominating role in the study.

Several fixed point theorems are discussed, starting from the contraction principle, Brouwer fixed-point theorem on finite dimensional space and ending on Schauder fixed point theorem. Their applications are illustrated by examples. In the next section we develop calculus on Banach space. This is a huge topic which has been split into different branches of mathematics such as the calculus of variations, optimization theory, control theory, etc.

# UNIT 8: HILBERT SPACE

**STRUCTURE**



## 8.0 OBJECTIVE

Understand the concept of Inner Product and relationship between inner product and norm, Enumerate the concept of Orthogonal Decomposition Understand Complete Orthonormal Sets

## 8.1 INTRODUCTION

The Euclidean norm is special among all norms defined in R*n* for being induced by the Euclidean inner product (the dot product). A Hilbert space is a Banach space whose norm is induced by an inner product. An inner product enables one to define orthogonality, which in turns leads to far reaching conclusion on the structure of a Hilbert space. In particular, we show that there is always a complete orthonormal set, a substitute for a Hamel basis, in a Hilbert space. It is a natural, infinite dimensional analog of an orthonormal basis in a finite dimensional vector space. We conclude with a theorem which asserts that any infinite dimensional separable Hilbert space is "isometric" to `2. Thus once again the cardinality of the basis alone is sufficient to distinguish separable Hilbert spaces.

David Hilbert was old and partly deaf in the nineteen thirties. Yet being a diligent man, he still attended seminars, usually accompanied by his assistant Richard Courant. One day a visitor was talking on his new findings in linear operators on Hilbert spaces. The professor was puzzled first. Soon he grew impatient and finally he turned to Courant. "Richard, what is a Hilbert space?" he asked loudly.

## 8.2 INNER PRODUCT

An **inner product** is a map: $X \times X \mapsto \mathbb{F}$ for a vector space $X$ over $\mathbb{F}$ satisfying

(P1) $\langle \alpha_1 x_1 + \alpha_2 x_2, y\rangle = \alpha_1\langle x_1, y\rangle + \alpha_2\langle x_2, y\rangle$,

(P2) $\overline{\langle x, y\rangle} = \langle y, x\rangle$,

(P3) $\langle x, x\rangle \geq 0$ and "=" if and only if $x = 0$.

The pair $(X, \langle\cdot,\cdot\rangle)$ is called an **inner product space**. Note that (P1) and (P2) imply

$$\langle x, \alpha_1 y_1 + \alpha_2 y_2\rangle = \overline{\alpha_1}\langle x, y_1\rangle + \overline{\alpha_2}\langle x, y_2\rangle.$$

**Example:** In $\mathbb{F}^n$ define the product

$$\langle x, y\rangle = \sum_{k=1}^{n} x_k \bar{y}_k.$$

It makes $(\mathbb{F}^n, \langle\cdot,\cdot\rangle)$ into an inner product space. It is called the Euclidean space when F*n* = R*n* and the unitary space when $\mathbb{F}^n = \mathbb{C}^n$.

**Example:** $\ell^2 = \{x = (x1, x2, x3, \cdots) : x_k \in \mathbb{F}$

$$\langle x, y\rangle = \sum_{k=1}^{\infty} x_k \bar{y}_k.$$

We should keep in mind that this product is finite is a consequence of Cauchy-Schwarz inequality. A variant of this space is the space of bi-sequences:

$$\ell^2(\mathbb{Z}) = \{x = (\cdots, x_{-1}, x_0, x_1, x_2, \cdots) : \quad \sum_{-\infty}^{\infty} |x_k|^2 < \infty\}$$

under the product

$$\langle x, y\rangle = \sum_{-\infty}^{\infty} x_k \bar{y}_k.$$

**Example:** Recall that $L^2(a, b)$ is the completion of $C[a, b]$ under the $L^2$-norm. On $C[a, b]$ the $L^2$-product

$$\langle f, g\rangle = \int f(x)\overline{g(x)}x,$$

Defines an inner product in $C[a, b]$. It is not hard to show that it has an extension to $L2\ (a, b)$. See next section for more details.

**Example:** Any subspace of an inner product space is an inner product space under the same product.

On F2 and the $\ell^2$-space there are Cauchy-Schwarz inequality. In fact, the most general setting for the Cauchy-Schwarz inequality is an inner product space. In the following we establish this inequality and use it to introduce the angle between two non-zero vectors and the concept of orthogonality.

**Proposition8.2.1.** For any x and y in an inner product space (X, ⟨·,·⟩),

Moreover, equality holds in this inequality if and only if x and y are linearly dependent.

***Proof***. The inequality is trivial when $x$ or $y$ is a zero-vector, so let's assume both $x$ and $y$ are non-zero. When the field is complex, let $\theta$ be a number satisfying $hx, yi = e^{i\theta}|hx, yi|$. Then $\langle x, z\rangle = |\langle x, y\rangle|$ where $z = e^{-i\theta}$ $y$ is a non-negative number. When the field is real, no need to do this as $\langle x, y\rangle$ is already real.

Just take *z* to be *y*. By (P3)

$$0 \leq \langle x - \alpha z, x - \alpha z\rangle = \langle x, x\rangle - 2\alpha\langle x, z\rangle + \alpha^2\langle z, z\rangle.$$

This is a quadratic equation with real coefficients in $\alpha$. Since it is always nonnegative, its discriminant is non-positive. In other words the inequality follows.

$$\langle x, z\rangle^2 - \langle x, x\rangle\langle z, z\rangle \leq 0,$$

When *x* and *y* are linearly dependent, there is some $\alpha \in$ F, $x - \alpha y = 0$. Plugging in the inequality. we readily see that equality holds. On the other hand, when $\langle x, x\rangle \langle y, y\rangle = |\langle x, y\rangle|^2$, we can take $\alpha = \langle x, y\rangle / \langle y, y\rangle$ in$\langle x - \alpha y, x - \alpha y\rangle$. By a direct computation, $0 = \langle x - \alpha y, x - \alpha y\rangle$..
By (P3), $x = \alpha y$.
It follows from this inequality that

$$\frac{|\langle x, y\rangle|}{\sqrt{\langle x, x\rangle}\sqrt{\langle y, y\rangle}} \leq 1.$$

For any two nonzero vectors *x* and *y* there is a unique $\theta \in [0, \pi]$ (the "angle" between *x* and *y*) satisfying

$$\theta = \cos^{-1}\frac{\langle x, y\rangle}{\sqrt{\langle x, x\rangle}\sqrt{\langle y, y\rangle}} \in [0, \pi].$$

Any two vectors *x* and *y* are **orthogonal** if $\langle x, y\rangle = 0$. The zero vector is orthogonal to all vectors.

## 8.3 INNER PRODUCT AND NORM

There is a norm canonically associated to an inner product. Indeed, the function $||\cdot|| : (X, \langle\cdot,\cdot\rangle) \mapsto [0, \infty)$
given by

$$\|x\| = \sqrt{\langle x, x\rangle}$$

defines a norm on *X*. To verify this, we only need to prove the triangle inequality since it is evident for the other two axioms. For $x, y \in X$

$$\begin{aligned} \|x+y\|^2 &= \|x\|^2 + \langle x,y\rangle + \langle y,x\rangle + \|y\|^2 \\ &\le \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \quad \text{(Cauchy-Schwarz inequality)} \\ &= (\|x\| + \|y\|)^2. \end{aligned}$$

Notions such as limits, convergence, open/closed sets and continuity in an inner product space will be referred to this induced norm. In particular, we have

**Proposition 8.3.1.** The inner product X × X ↦ [0, ∞) is a continuous function.

*Proof.* For $x_n \to x$ and $y_n \to y$, we want to show that $\langle x_n, y_n\rangle \to \langle x, y\rangle$. First of all, let $n_0$ be a positive number satisfying $||y_n - y|| \le 1$, $\forall\ n \ge n0$. Then $||y_n|| \le 1 + ||y||$ and we have as $n \to \infty$.

$$\begin{aligned} |\langle x_n, y_n\rangle - \langle x, y\rangle| &= |\langle x_n - x, y_n\rangle + \langle x, y_n - y\rangle| \\ &\le \|x_n - x\|\|y_n\| + \|x\|\|y_n - y\| \\ &\le \|x_n - x\|(1 + \|y\|) + \|x\|\|y_n - y\| \\ &\longrightarrow 0 \end{aligned}$$

A complete inner product space is called a **Hilbert space**. As any closed subspace of a Banach space is complete, any closed subspace of a Hilbert space is also a Hilbert space. Products of Hilbert spaces are Hilbert spaces. Moreover, the quotient space of a Hilbert space over a closed subspace is again a Hilbert space. For completion of an inner product space we have the following rather evident result.

**Proposition 8.3.2.** Let (X, e k·e k) be the completion of (X, k·k) where k·k is induced from an inner product⟨·,·⟩. Then there exists a complete inner product on ⟨·,·̃⟩. Which extends ⟨·,·⟩ and induces ||.|̃|

when the reader runs through his/her list of normed spaces, he/she will find that there are far more Banach spaces than Hilbert spaces. However, one may wonder these Banach spaces are also Hilbert spaces, whose inner products are just too obscure to write down. A natural question arises: How can we decide which norm is induced by an inner product and which one is not? The answer to this question lies on a simple property—the parallelogram identity.

**Proposition 8.3.3 (Parallelogram Identity).** *For any x, y in* $(X, \langle\cdot,\cdot\rangle)$,

*Proof.* Expand and add up.

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

$$\|x \pm y\|^2 = \|x\|^2 \pm \langle x, y\rangle \pm \langle y, x\rangle + \|y\|^2$$

This rule, which involves only the norm but not the inner product, gives a necessary condition for a norm to be induced by an inner product

As application we first show that the $\|\cdot\|_p$-norm on $\mathbb{F}^n$ $(n \geq 2)$ is induced from an inner product if and only if $p = 2$. Take $x = (1, 1, 0, \cdots, 0)$ and $y = (1, -1, 0, \cdots, 0)$ in $\mathbb{F}^n$. We have $\|x\|_p = \|y\|_p = 2^{1/p}$ and $\|x + y\|_p = \|x - y\|_p = 2$. If $\|\cdot\|_p$ is induced from an inner product, Proposition 8.2.3 asserts which holds only if $p = 2$.

$$\|x+y\|_p^2 + \|x-y\|_p^2 = 8 = 2(\|x\|_p^2 + \|y\|_p^2) = 2^{\frac{2}{p}+2}$$

Similarly, $C[0, 1]$ does not come from an inner product. We take $f(x) = 1$ and $g(x) = x$. Then

$$\|f\|_\infty = \|g\|_\infty = 1,\ \|f+g\|_\infty = 2 \text{ and } \|f-g\|_\infty = 1.$$

Then

$$\|f+g\|_\infty^2 + \|f-g\|_\infty^2 = 5 \neq 2(\|f\|_\infty^2 + \|g\|_\infty^2) = 4.$$

**Proposition 8.2.4.** (a) For every x, y in a real inner product space X, we have

$$\langle x, y\rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2).$$

(b) On a real normed space (X, $\|\cdot\|$), the above identity defines an inner product on X if and only if the parallelogram identity holds.

The identity in (a) is called the **polarization identity**. It shows how one can recover the inner product from the norm in an inner product space.

Proof. (a) We have By subtracting, we get the polarization formula.

$$\begin{aligned}\|x \pm y\|^2 &= \|x\|^2 \pm \langle x, y\rangle \pm \langle y, x\rangle + \|y\|^2 \\ &= \|x\|^2 \pm 2\langle x, y\rangle + \|y\|^2.\end{aligned}$$

(b) In view of Proposition 8.2.3, it remain to verify that the polarization identity defines an inner product under the validity of the parallelogram identity. In fact, (P2) and (P3) are immediate. We only need to prove (P1). By the parallelogram identity,

$$2(||x \pm z||^2 + ||y||^2) = ||(x \pm z) + y||^2 + ||(x \pm z) - y||^2.$$

By subtracting

$$2(||x + z||^2 - ||x - z||^2) = ||(x + y) + z||^2 - ||(x + y) - z||^2 + ||(x - y) + z||^2 - ||(x - y) - z||^2.$$

In terms of $\langle\cdot,\cdot\rangle$ , we have $\langle x + y, z\rangle + \langle x - y, z\rangle = 2\langle x, z\rangle$ for all $x, y, z \in X$. Replacing $x, y$ by $(x + y)/2$ and $(x - y)/2$ respectively, we have

$$\langle x, z\rangle + \langle y, z\rangle = 2\langle \frac{x+y}{2}, z\rangle, \quad \text{for all } x, y, z \in X.$$

Letting $y = 0$, $\langle x, z\rangle = 2hx/2, zi$ for all $x$, $z$. It follows that

$$\langle x + y, z\rangle = 2\langle \frac{x+y}{2}, z\rangle = \langle x, z\rangle + \langle y, z\rangle,$$

that is, $\langle\cdot,\cdot\rangle$ is additive in the first component. Next, we show that $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$, for all $\alpha \in$ R. We observe that by induction and 2 $\langle x, y\rangle = \langle 2x, y\rangle$ we can show $n\langle x, y\rangle = \langle nx, y\rangle$ for all $n \in$ N. Using $\langle x, y\rangle + \langle -x, y\rangle = \langle 0, y\rangle = 0$ we deduce $n\langle x, y\rangle = \langle nx, y\rangle$ for all $n \in$ Z. Then $m$

$\langle x/m, y\rangle = \langle x/m, y\rangle + \cdots + \langle x/m, y\rangle$ ($m$ times)$= \langle x, y\rangle$ implies $1/m$ $\langle x/m, y\rangle = \langle x, y\rangle$. So, $\langle nx/m, y\rangle = 1/m \langle nx, y\rangle = n/m \langle x, y\rangle$ for any rational $n/m$. By continuity of the norm, $\langle \alpha x, y \rangle = \alpha \langle x, y\rangle$,

We have the following corresponding result when F = C, whose proof may be deduced from the real case.

**Proposition 8.3.5.** *(*a) For any x, y in a complex inner product space X, we have the polarization identities

$$Re\langle x, y\rangle = \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2),$$

$$Im\langle x, y\rangle = \frac{1}{4}(\|x + iy\|^2 - \|x - iy\|^2).$$

(b) On a complex normed space X the polarization identities define an inner product on X which induces its norm if and only if the parallelogram identity holds.

**CHECK YOUR PROGRESS**

1. What is inner product space?

2. Explain **Parallelogram Identity**

## 8.4 ORTHOGONAL DECOMPOSITION

Aside from finite dimensional subspaces,the problem does not always have a positive solution. Nevertheless, with the help of orthogonality, we show in this section that for a Hilbert space this problem always has a unique solution. An immediate consequence is the existence of complementary subspaces, a property which is not necessarily valid for Banach spaces. In fact, our result extends from subspaces to convex subsets.

**Theorem 8.4.1.** *Let K be a closed, convex subset in the Hilbert space X and x0 ∈ X \ K. There exists a unique point x∗ ∈ K such that*

$$\|x_0 - x^*\| = \inf_{x\in K} \|x_0 - x\|$$

Proof. Let $\{x_n\}$ be a minimizing sequence in K, in other words,

$$\|x_0 - x_n\| \to d = \inf_{x\in K} \|x_0 - x\|.$$

We claim that $\{x_n\}$ is a Cauchy sequence. For, by parallelogram identity,

$$\begin{aligned} \|x_n - x_m\|^2 &= \|x_n - x_0 - (x_m - x_0)\|^2 \\ &= -\|x_n - x_0 + x_m - x_0\|^2 + 2(\|x_n - x_0\|^2 + \|x_m - x_0\|^2) \\ &= -4\|\frac{x_n + x_m}{2} - x_0\|^2 + 2(\|x_n - x_0\|^2 + \|x_m - x_0\|^2) \\ &\leq -4d^2 + 2(\|x_n - x_0\|^2 + \|x_m - x_0\|^2) \\ &\to 0 \end{aligned}$$

as $n, m \to \infty$. Note that $(x_n + x_m)/2 \in K$ by convexity. By the completeness of $X$ and the closedness of $K$, $x* = \lim n\to\infty\ x_n \in K$. As the norm is a continuous function, we have $d = kx0 - x*k$. Suppose $x0 \in K$ also minimizes the distance. Then

$$\begin{aligned} \|x^* - x'\|^2 &\leq -4\|\frac{x^* + x'}{2} - x_0\|^2 + 2(\|x^* - x_0\|^2 + \|x' - x_0\|^2) \\ &\leq -4d^2 + 4d^2 = 0, \end{aligned}$$

that's, $x' = x^*$.

This theorem plays a fundamental role in convex analysis. But here we only consider the special case when $K$ is a closed subspace. More can be said about the best approximation in this case.

**Theorem 8.4.2 (Best Approximation).** Let Y be a closed subspace of a Hilbert space X and $x_0$ a point lying outside Y . The point $y_0$ which minimizes the distance between $x_0$ and Y satisfies

$$\langle x_0 - z, y\rangle = 0, \qquad \forall y \in Y,$$

*at z =* $y_0$. *Conversely, if z* $\in Y$ *satisfies this condition, then z must be* $y0$.
*When this holds*

$$||x_0 - y_0||^2 + ||y_0||^2 = ||x_0||^2, \qquad (1)$$

***Proof.*** When $y_0$ minimizes $||y - x_0||$ among all $y$, it also minimizes $||y - x_0||^2$. For any $y \in Y$, $y_0 + \varepsilon y \in Y$, so the function

$$\phi(\varepsilon) = ||x_0 - y_0 - \varepsilon y||^2$$

attains a minimum at $\varepsilon = 0$. By expanding, we have

$$\varphi(\varepsilon) = ||x_0 - y_0||^2 - \varepsilon\langle x_0 - y_0, y\rangle - \varepsilon\langle y, x_0 - y_0\rangle + \varepsilon^2||y||^2.$$

Clearly $0 = \varphi'(0)$ implies

$$Re\,\langle x_0 - y_0, y\rangle = 0.$$

Replacing $y$ by $iy$, $Im\,\langle x_0 - y_0, y\rangle = 0$.
Conversely, if $\langle x_0 - y_0, y\rangle = 0$ for all $y$ in $Y$, we have

$$\begin{aligned} ||x_0 - y||^2 &= ||x_0 - y_0 + y - y_0||^2 \\ &= ||x_0 - y_0||^2 + ||y - y_0||^2 \\ &\geq ||x_0 - y_0||^2, \end{aligned}$$

which shows that $y_0$ minimizes $d(x_0, Y)$.

Finally, let $y1$ also minimize the distance. By the above characterization, $hx0 - y1, yi = 0$ on $Y$. It follows that

$$\langle y_0 - y_1, y\rangle = \langle x_0 - y_1, y\rangle - \langle x_0 - y_0, y\rangle = 0.$$

This theorem has the following geometric meaning. For $x_0$ outside $Y$, the projection point $y_0$ is the unique point on $Y$ so that $\Delta O x_0 y_0$ forms a perpendicular triangle.

For any closed subspace $Y$ in a Hilbert space, the **projection operator** of $X$ onto $Y$ is given by

$$Px_0 = \begin{cases} y_0, & \text{if } x_0 \in X \setminus Y \\ x_0, & \text{if } x_0 \in Y \end{cases}$$

We have been calling $Px$ the best approximation of $x$ in $Y$. Now we may also it the **orthogonal projection** of $x$ on $Y$. It is easy to check that $P \in B(X, Y)$, $P^2 = P$ and $||P|| = 1$. For instance, to show that $P$ is linear, we just have to verify the obvious identity $\langle \alpha x_1 + \beta x_2 - (\alpha P x_1 + \beta P x_2), y \rangle = 0$ on $Y$. For then it follows from the above characterization that $P(\alpha x_1 + \beta x_2) = \alpha P x_1 + \beta P x_2$.

We also note that a more general characterization holds: For any $x$ in $X$, not only those in $Y$, $z = Px$ if and only if $z$ satisfies $\langle x - z, y \rangle = 0$ on $Y$. We will discuss two consequences of the best approximation theorem. The first is the self-duality property of the Hilbert space.

To each $z$ in the Hilbert space $X$ we associate a bounded linear functional $\Lambda z$ given by $\Lambda z x = \langle x, z \rangle$. It is routine to verify that $\Lambda z$ belongs to $X'$ with operator norm $||\Lambda z|| = ||z||$. The mapping $\Phi$ defined by mapping $z$ to $\Lambda z$ sets up a sesquilinear map from $X$ to $X0$. A map $T$ is **sesquilinear** if $T(\alpha x_1 + \beta x_2) = \bar{\alpha} T x_1 + \bar{\beta} T x_2$. Sesquilinear and linear are the same when the field is real, and they are different when the field is complex. The following Frech´et-Riesz theorem shows that $\Phi$ is surjective, so it is a normpreserving, sesquilinear isomorphism between $X$ and $X0$. A Hilbert space is self-dual in the sense that every bounded linear functional on it can be identified with a unique point in the space itself.

**Theorem 8.4.3.** Let X be a Hilbert space. For every Λ in X', there exists a unique z in X such that Λ z = Λ and ||z||= ||Λ||.

Next, we consider direct sum decomposition in a Hilbert space. Recall that a direct sum decomposition of a vector space, $X = X_1 \oplus X_2$, where $X_1$ and $X_2$ are two subspaces, means every vector $x$ can be expressed as the sum of one vector $x_1$ from $X_1$ and the other $x_2$ from $X_2$ in a unique way. From the uniqueness of the representation one can show that the maps $x \mapsto x_1$ and $x \mapsto x_2$ are linear maps from $X$ onto $X_1$ and $X_2$ respectively. They are called projection maps associated to the direct sum $X_1 \oplus X_2$.

Direct sum decomposition is clearly useful in the study of vector spaces since it breaks down the space to two smaller (and hence simpler) subspaces. When the space is normed, it is desirable to ensure that such decomposition respects the topology in some sense. Thus we may introduce the definition that the direct sum is a "topological direct sum" if the projection maps: $x \mapsto x1$ and $x \mapsto x2$ are bounded from $X$ to $Xi$, $i$ = 1, 2. When the space is complete, certainly we would like our decomposition to break into Banach spaces. We prefer $Xi$, $i$ = 1, 2, to be closed subspaces. An advantage of this assumption is that the projections are automatically bounded, as a direct consequence of the closed graph theorem, so any direct sum decomposition of a Banach space into closed subspaces is topological.

We now are left with question: Given any closed subspace $X1$ of a Banach space $X$, can we find a closed subspace $X2$ such that $X = X1 \oplus X2$? Unfortunately, except when $X1$ is of finite dimension, a complementary closed space $X2$ does not always exist. However, this is always true for Hilbert spaces. In fact, a deep theorem asserts that if a Banach space possesses the property that any closed subspace has a topological complement, then its norm must be equivalent to a norm induced by a complete inner product.

In fact, for any closed, proper subspace $X1$, we define its "orthogonal subspace" $X_1^\perp$ to be

$$X_1^\perp = \{x \in X : \langle x, x_1 \rangle = 0, \text{ for all } x_1 \in X_1\}$$

It is clear that $X_1^\perp$ is a closed subspace. (According to Riesz-Frechet theorem, $X_1^\perp$ is the annihilator of$X_1$.) Thus we have the decomposition

$x = P x + (x - P x) \in X1 + X_1^\perp$ where $P$ is the orthogonal projection operator on $X_1$. We claim that this is a direct sum. For, if $x_0 \in X1 \cap X_1^\perp$

Notes

then $\langle x_0, x_1 \rangle = 0$, for all $x_1 \in X_1$. As $x0$ also belongs to $X_1$, taking $x_1 = x_0$, we get $||x_0||^2 = \langle x_0, x_0 \rangle = 0$. Hence $X_1 \cap X_1^{\perp} = \{0\}$. Moreover, we observe that the bounded linear operator $P$ and $I - P$ are precisely the projection maps of the direct sum $X_1 \oplus X_1^{\perp}$.We have proved the following theorem on the orthogonal decomposition in a Hilbert space.

**Theorem 8.4.4.** For every closed subspace X1 of a Hilbert space X, X = X1 $\oplus$ $X_1^{\perp}$.Moreover, the projection operator P : X $\mapsto$ $X_1$ maps x to Px which is the unique point in $X_1$ satisfying ||x − P x|| = d(x, X1) and the projection Q : X $\mapsto$ $X_1^{\perp}$ is given by Qx = x − P x.

# 8.5 COMPLETE ORTHONORMAL SETS

We start by considering the following question: How can we determine $P x_0$ when $x_0$ and the subspace $Y$ are given? It is helpful to examine this question when $X$ is the n-dimensional Euclidean space. Let $\{x1, \cdots, xm\}$ be a basis of $Y$. Any projection $y0$ has the expression $y0 = \sum_{k=1}^{m} \alpha_k x_k$. From Theorem 8.3.2 we have $\langle y_0 - x_0, x_k \rangle = 0$ for each $k = 1, \cdots, m$. It amounts to a linear system for the unknown $\alpha'_j s$:

$$\sum_k \langle x_j, x_k \rangle \alpha_k = \langle x_j, x_0 \rangle.$$

The system assumes a simple form when $\{x_k\}$ forms an orthonormal set. We immediately solve this system to get $y_0 = \sum_k \langle x_0, x_k \rangle$. This example suggests it is better to consider orthonormal spanning sets in $Y$.

**Lemma 8.5.1 (Bessel's Inequality).** Let S be an orthonormal set in the Hilbert space X. Then for each x ∈ X, $\langle x, x_\alpha \rangle = 0$ except for at most countably many $x_\alpha \in B$. Moreover, for any sequence $\{\alpha_k\}$ from the index set B

$$\sum_k |\langle x, x_{\alpha_k} \rangle|^2 \leq \|x\|^2. \qquad (2)$$

*Proof.* Step 1: Let $\{x_k\}_1^N$be a finite orthonormal set. For $x \in X$, we claim

$$\sum_{k=1}^{N} |\langle x, x_k \rangle|^2 \leq \|x\|^2. \qquad (3)$$

For, let $y = \sum_{k=1}^{N} \langle x, x_k \rangle x_k$. Then $\langle x - y, x_k \rangle = 0$, for all $k = 1, \cdots, N$, implies that $y$ is the orthogonal projection of $x$ onto the space $hx1, \cdots,$ $xNi$. As $||y||^2 = \sum_{k=1}^{N} |\langle x, x_k \rangle|^2$

$$\sum_{k=1}^{N} |\langle x, x_k \rangle|^2 = \|y\|^2 = \|x\|^2 - \|x - y\|^2 \leq \|x\|^2,$$

Step 2: Let $x \in X$ and $l$ a natural number. We claim that the subset $Sl$ of $S$

$$S_l = \{x : |\langle x, x_\alpha \rangle| \geq \frac{1}{l}\}$$

is a finite set. For, picking $x_{\alpha_1}, \cdots, x_{\alpha_N}$ many vectors from $S_l$ and applying Step 1, we get

$$\|x\|^2 \geq \sum_{k=1}^{N} |\langle x, x_{\alpha_k} \rangle|^2 \geq \frac{N}{l^2}.$$

It follows that the cardinality of $S_l$ cannot exceed $||x||^2 l^2$.

Step 3: Only countably many terms $\langle x, x_\alpha \rangle$are non-zero. Let $S_x$ be the subset of $S$ consisting of all $x_\alpha$'s such that $\langle x, x_\alpha \rangle$is non-zero. We have the decomposition Since each $S_l$ is a finite set, $S_x$ is countable.

$$S_x = \bigcup_{1}^{\infty} S_l.$$

Now the Bessel's inequality follows from passing to infinity .

Now we can give an answer to the question posed in the beginning of this section.

**Theorem 8.5.2.** Let Y be a closed subspace in the Hilbert space X. Suppose that S is an orthonormal subset of Y whose linear span is dense in Y. Then for each x, its orthogonal projection on Y is given by $\sum_{k=1} \langle x, x_k \rangle x_k$. where $\{x_k\}$ is any ordering of all those $x\alpha$ in S with non-zero $\langle x, x_\alpha \rangle$.

*Proof.* First of all, we need to verify that the sum $\sum_k \langle x, x_k \rangle\, x_k$ is convergent. By the completeness of *Y* it suffices to show that *{yn}* ≡ $\{\sum_k^n \langle x, x_k \rangle\, x_k.\ \}$ is a Cauchy sequence. Indeed, we have

$$
\begin{aligned}
\|y_n - y_m\|^2 &= \|\sum_{m+1}^{n} \langle x, x_k \rangle x_k\|^2 \\
&= \sum_{m+1}^{n} |\langle x, x_k \rangle|^2.
\end{aligned}
$$

By the Bessel's inequality it is clear that the right hand side of this inequality tends to zero as $n,\ m \to \infty$. Therefore, $\{y_n\}$ is a Cauchy sequence. Now using the characterization of the orthogonal projection stated in Theorem 8.3.2 and the continuity of the inner product, we conclude the proof of this theorem.

After proving that the sum $\sum_k \langle x, x_k \rangle\, x_k$ is the projection of *x* on *Y* , we see that this summation is independent of the ordering of those non-zero$\langle x, x_\alpha \rangle$ .(You may also deduce this result by recalling rearrangement does not change the limit of an absolutely convergent series.) Therefore we can comfortably write it in the form $\sum_\alpha \langle x, x_\alpha \rangle\, x_\alpha$without causing any confusion.

The discussion so far motivates us to introduce a more replacement of the Hamel basis for Hilbert spaces.

A subset *B* in a Hilbert space is called a **complete orthonormal set** if it satisfies (a) it is an orthonormal set, that is, for all $x \neq y \in B$,$\langle x, y \rangle$= 0, and $||x||$ = 1, and (b) $\langle B \rangle$ = *X*. The conditions are different from those for a basis. In contrast, for a basis *B* we require (a)' all vectors in *B* is linearly independent, and (b)' *hBi* = *X*. It is an exercise to show that (a) implies (a)', but (b) is weaker than (b)'. Some authors use the terminology "an orthonormal basis" instead of "a complete orthonormal set". We prefer to use the latter.

**Theorem 8.5.3 .** Every non-zero Hilbert space admits a complete orthonormal set.

*Proof.* Let $\mathcal{F}$ be the collection of all orthonormal sets in *X*. Clearly *F* is non-empty and has a partial order by set inclusion. For any chain $\mathcal{C}$ in $\mathcal{F}$ , clearly

is an upper bound of $\mathcal{C}$ . By Zorn's lemma, $\mathcal{F}$ has a maximal element *B*. We claim that *B* is a complete orthonormal set. First of all, *B* consists of normalized vectors orthogonal to each other, so (a) holds. To prove (b), let's assume there is some *z* not in $\langle B \rangle$. By orthogonal decomposition, $z'$ $= (z - Pz)/||z - Pz||$ where *P* is the projection operator on $\langle B \rangle$ is a unit vector perpendicular to $\langle B \rangle$. It follows that $B \cup \{_{z0}\} \in \mathcal{F}$ , contradicting the maximality of *B*.
We end this section with some criteria for a complete orthonormal set.

**Theorem 8.5.4.** Let B be an orthonormal set in the Hilbert space X. The followings are equivalent:

*(a)* *B is a complete orthonormal set,*

*(b)* $x = \sum_{x_\alpha \in B} \langle x, x_\alpha \rangle x_\alpha$ *holds for all* $x \in X$,

*(c)* $\|x\|^2 = \sum_{x_\alpha \in B} |\langle x, x_\alpha \rangle|^2$ *holds,*

*(d)* $\langle x, x_\alpha \rangle = 0$, *for all* $x_\alpha \in B$ *implies that* $x = 0$.

(c) is called the (**Parseval's identity**). In other words, the Bessel's inequality holds on every orthonormal set, but the Parseval's identity holds only when the set is complete.
*Proof.* (a)⇒(b): When $\langle B \rangle = X$, the orthogonal projection beomes the identity map, so (b) holds by Theorem 8.4.2.
(b)⇒(c)

$$\|x\|^2 - \sum_{k=1}^{n} |\langle x, x_k \rangle|^2 = \|x - \sum_{k=1}^{n} \langle x, x_k \rangle x_k\|^2 \to 0,$$

as $n \to \infty$, by Theorem 8.4.2.
(c)⇒(d): Obvious.

(d)⇒(a): Suppose on the contrary $\langle B\rangle$ is strictly contained in $X$. We can find a non-zero $x_0 \in X\backslash\langle B\rangle$ such that $\langle x_0, x_\alpha\rangle = 0$, for all $x_\alpha \in B$. However, this is impossible by (d).

**CHECK YOUR PROGRESS**

3.Explain: Every non-zero Hilbert space admits a complete orthonormal set

4. State projection operator

# 8.6 LET'S SUM UP

Many of the applications of Hilbert spaces exploit the fact that Hilbert spaces support generalizations of simple geometric concepts like projection and change of basis from their usual finite dimensional setting. In particular, the spectral theory of continuous self-adjoint linear operators on a Hilbert space generalizes the usual spectral decomposition of a matrix, and this often plays a major role in applications of the theory to other areas of mathematics and physics.

# 8.7 KEYWORDS

**Normalized vectors** - The **normalized vector** of is a **vector** in the same direction but with norm (length) 1

**Identity** - an **identity** is an equality relating one **mathematical** expression A to another **mathematical** expression B, such that A and B (which might contain some variables) produce the same value for all values of the variables within a certain range of validity.

**Defines** - being **defined** as , where both and are valid syntactic strings, **means** that can be replaced by wherever it occurs without affecting the **meaning** of the sentence.

**Validity** - The **validity** of a logical argument refers to whether or not the conclusion follows logically from the premises, i.e., whether it is possible to deduce the conclusion from the premises and the allowable syllogisms of the logical system being used

## 8.8 QUESTIONS FOR REVIEW

1. Establish the identity

$$\|z-x\|^2+\|z-y\|^2=\frac{1}{2}\|x-y\|^2+2\|z-\frac{1}{2}(x+y)\|$$

for *x*, *y*, and *z* in an inner product space.

2. Show that (*C*[0, 1], *k* · *kp*) is not induced from any inner product for *p* ∈ [1, ∞] \ {2}

3. Let *X* be a Hilbert space. Show that if *{xn} weakly converges* to *x*, that's, Λ*xn* → Λ*x*, ∀Λ ∈ *X0*, then $x_n \to x$ provided $||x_n|| \to ||x||$.

4. Find the orthogonal projection of (or the best approximation to) the function *f* onto the subspace spanned by cos *x* and 1 − 2*x* in $L^2(-\pi, \pi)$ where *f* is (a) *ex* and (b) cos 6*x*.

## 8.9 SUGGESTED READINGS

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## 8.10 ANSWER TO CHECK YOUR PROGRESS

1. Provide statement of proposition and proof – 8.2.1

2. Provide statement and proof – 8.3.3

3. Refer section below – 8.4.2

4. Provide proof – 8.5.3

# UNIT 9: STRUCTURE THEOREM & COMPACT, SELF-ADJOINT OPERATOR IN HILBERT SPACE

**STRUCTURE**



## 9.0 OBJECTIVE

Understand the concept of Structure Theorem

Comprehend the Adjoint Operators

Understand the Compact, Self-Adjoint Operators

## 9.1 INTRODUCTION

Recall that in linear algebra we showed that two finite dimensional vector spaces are linearly isomorphic if and only if they have the same dimension. That means there is only one invariant, the dimension, to distinguish vector spaces. A similar result holds in a (separable) Hilbert space. We prove in below that every separable Hilbert space has a countable complete orthonormal set. Consequently separable Hilbert spaces are distinguished by their cardinality.

Notes

To have a taste of the richness of operator theory, here we cast our attention on a special class of bounded linear operators, namely, compact, self-adjoint ones in Hilbert spaces. Our main result is a structural theorem stating that the eigenvectors of a compact, self-adjoint operators form a complete orthonormal set. This is an infinite dimensional generalization of the theorem of reduction to principal axes for self-adjoint matrices in linear algebra. Compact, self-adjoint operators come up naturally in differential and integral equations. In the last section we show how it is applied to the boundary value problems of second order ordinary differential equations.

# 9.2 STRUCTURE THEOREM:

**Proposition 9.2.1.** A Hilbert space has a countable complete orthonormal set if and only if it is separable in its induced metric.

*Proof.* Let $B = \{x_k\}_1^\infty$ be a complete orthonormal set of *X*. By definition, the set $\langle B \rangle$ is dense in *X*. However, consider the subset $S = \{x \in \langle B \rangle : x$ is a linear combination of *B* with coefficients in Q or Q + *i*Q depending on F = R or C$\}$. It is clear that $S = \langle B \rangle = X$.
On the other hand, let *C* be a countable, dense subset of *X*. We can write it as a sequence $\{x1, x2, x3, \cdots\}$. Step by step we can throw away vectors which are linearly dependent of t he previous ones to get a subset $\{y1, y2, y3, \cdots\}$ which consists of linearly independent vectors and yet still spans *X*. Now, apply the Gram-Schmidt process to this subset to obtain an orthonormal set $\{z1, z2, z3, \cdots\}$. From construction we have that $\langle \{z1, z2, z3, \cdots\} \rangle = \langle \{y1, y2, y3, \cdots\} \rangle$ so $\{z1, z2, z3, \cdots\}$ is a complete orthonormal set.

**Theorem 9.2.2.** Every infinite dimensional separable Hilbert space X is the same as $\ell^2$. More precisely, there exists an inner-product preserving linear isomorphism Φ from X to $\ell^2$.
*Proof.* Pick a complete orthonormal set $\{x_k\}_1^\infty$ of *X* whose existence is guaranteed by Proposition 9.1.2.Then for every $x \in X$, we have $x = \sum \langle x, x_k \rangle x_k$. Define the map $\Phi : X \mapsto \ell^2$by $\Phi(x) = (a1, a2, \cdots)$ where $ak = \langle x, x_k \rangle$. By Theorem 9.1.2 we know that Φ is a norm-preserving linear

map from $X$ to $\ell^2$. It is also onto. For, let $\{a_k\}$ be an $l^2$-sequence. Define $y_n = \sum_{k=1}^{n} a_k x_k$ Using $||y_n - y_m||^2 = \sum_{m+1}^{n} a_k^2 \to 0$ as $n, m \to \infty$, $y_n$ converges in $X$ to $\sum_1^\infty a_k x_k$. Clearly, $\langle x, x_k \rangle = a_k$, so $\Phi$ is onto. Finally, it is also inner-product preserving by polarization.

We end this section with a famous example of a complete orthonormal set.

First, Let $L^2((-\pi, \pi))$ be the completion of $C([-\pi, \pi])$ under the $L^2$-product. For $f, g \in C([-\pi, \pi])$

over the complex field, the product is given by

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)\overline{g(x)} dx.$$

The set

$$B = \{\frac{1}{\sqrt{2\pi}} e^{inx}, n \in \mathbb{Z}\}$$

is a countable set consisting of orthonormal vectors:

$$\begin{aligned}
\langle \frac{1}{\sqrt{2\pi}} e^{inx}, \frac{1}{\sqrt{2\pi}} e^{imx} \rangle &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{inx} \overline{e^{imx}} dx \\
&= \frac{1}{2\pi} \frac{1}{i(m-n)} e^{i(n-m)x} |_{-\pi}^{\pi} \\
&= 0 \text{ if } n \neq m;
\end{aligned}$$

$$\langle \frac{1}{\sqrt{2\pi}} e^{inx}, \frac{1}{\sqrt{2\pi}} e^{imx} \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{inx} e^{-inx} = 1.$$

For $f \in L^2((-\pi, \pi))$, we define its Fourier series to be

$$\sum_n \langle f, \frac{1}{\sqrt{2\pi}} e^{iny} \rangle \frac{1}{\sqrt{2\pi}} e^{inx} = \sum_{n=-\infty}^{\infty} c_n e^{inx}, \quad c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{-iny}\, dy,$$

and write $f \sim$ P$n$ *cneinx*. We know from previous discussions that this series is a well-defined function in $L^2((-\pi, \pi))$. In fact, it is the orthogonal projection of $f$ onto the closed subspace $\langle B \rangle$. The completeness of $B$ is a standard result in Fourier Analysis. Here we give a quick proof by using Weierstrass' approximation theorem in the plane. That is, for any continuous function $f$ in the unit disk there exists $\{pn(z)\}$, where each $pn$ is a polynomial so that $\{pn\}$ tends to $f$ in supnorm.

Observe any $2\pi$-periodic function $f$ in $[-\pi, \pi]$ induces a function $g \in C(S1)$ where $S1 = \{e^{i\theta} : \theta \in [-\pi, \pi]\}$ is the unit circle in the plane by $g(e^{i\theta}) = f(\theta)$. Extend $g$ as a continuous function in the closed disc $D = \{(x1, x2) : x_1^2 + x_2^2 + \leq 1\}$ and denote it by the same $g$. For $\varepsilon > 0$, by Weierstrass' theorem there exists a polynomial $p(x1, x2)$ such that $||p - g||_{\infty,\bar{D}} < \varepsilon$. When restricted to $S^1$ we obtain

$$\|p(x_1, x_2) - g(x_1, x_2)\|_{\infty, S^1} < \varepsilon,$$

$$x_1 = \frac{1}{2}(e^{ix} + e^{-ix}), \ x_2 = \frac{1}{2i}(e^{ix} - e^{-ix}), \ x \in [-\pi, \pi].$$

As $p(x1, x2)$ (regarded as a function of $x$ in $[-\pi, \pi]$) is just a linear combination of functions in $B$, this shows that $\langle B \rangle$is dense in the subspace of periodic functions in $C([-\pi, \pi])$ under the sup-norm. As the sup-norm is stronger than the $L^2$-norm, $\langle B\rangle$ is also dense in this subspace in $L^2$-norm. Now, for any $L^2$-function $f$, we can find a continuous function $f_1 \in C[-\pi, \pi]$ such that $||f - f_1||^2 < \varepsilon$. By modifying the value of $f$ near endpoints we can find another continuous $f_2$, which is now periodic, $||f_1 - f_2|| < \varepsilon$.

Finally, there exists a trigonometric polynomial $p$ such that $||f_2 - p||^2 < \varepsilon$. All together we obtain $||f - p||^2 \leq ||f - f1||^2 + ||f_1 - f_2||^2 + ||f2 - p||^2 < 3\varepsilon$. We conclude that $B$ forms a complete orthonormalset in $L^2((-\pi, \pi))$. In particular, for every $L^2$-functions, its Fourier series converges to $f$ in $L^2$-norm,and the Parseval's identity holds.

$$\int_{-\pi}^{\pi} |f(x)|^2 dx = 2\pi \sum_{n=-\infty}^{\infty} |c_n|^2,$$

For a real function $f$, the Fourier series is usually expressed in real form,

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(y)\cos ny\, dy,\ n \geq 0, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(y)\sin ny\, dy,\ n \geq 1,$$

where

and you can write down the corresponding Parseval's identity.

## 9.3 ADJOINT OPERATORS

Let $T \in B(X1, X2)$ where $X1$ and $X2$ are Hilbert spaces over the same field. We construct a bounded linear operator called the adjoint of $T$, $T^*$, from $X2$ to $X1$ as follows. For any $y \in X2$, the map $x \mapsto \langle Tx, y\rangle_{X_2}$ is linear and bounded, and hence defines an element in $X_1'$. By self-duality there exists a unique x* in $X_1$ such that $\langle Tx, y\rangle_{X_2} = \langle Tx, x^*\rangle_{X_1}$. We define the **adjoint** of $T$ to be the map $T^*y = x^*$.
Then

$$\langle Tx, y\rangle_{X_2} = \langle x, T^*y\rangle_{X_1}, \text{ for all } x, y, \qquad (1)$$

holds. We shall drop the subscripts in the inner products.

**Proposition 9.3.1.** Let T be in $B(X_1, X_2)$ where $X_1$ and $X_2$ are Hilbert spaces. Then

*(1)* $(T^*)^* = T$,

*(2)* $T^* \in B(X_2, X_1)$, *and*

*(3)* $\|T^*\| = \|T\|$.

Proof. (1) is straightforward from definition. Next we verify linearity. For any y1, y2 and scalars α and β, by (1)

$$\begin{aligned}\langle x, T^*(\alpha y_1 + \beta y_2)\rangle &= \langle Tx, \alpha y_1 + \beta y_2\rangle \\ &= \overline{\alpha}\langle Tx, y_1\rangle + \overline{\beta}\langle Tx, y_2\rangle \\ &= \overline{\alpha}\langle x, T^*y_1\rangle + \overline{\beta}\langle x, T^*y_2\rangle \\ &= \langle x, \alpha T^*y_1 + \beta T^*y_2\rangle,\end{aligned}$$

so $T^*(\alpha y_1 + \beta y_2) = \alpha T^*y_1 + \beta T^*y_2$.

Finally, by self-duality,

$$\|T^*y\| = \sup_{x\neq 0} \frac{|\langle x, T^*y\rangle|}{\|x\|} = \sup_{x\neq 0} \frac{|\langle Tx, y\rangle|}{\|x\|} \leq \|T\|\|y\|,$$

so $||T^*|| \leq ||T||$. The reverse inequality follows from (1).

Other elementary properties of $T*$ are contained in the following proposition, whose proof is left to you.

**Proposition 9.3.2.** Let T, T1, and T2 ∈ B($X_1$, $X_2$) where X1 and X2 are Hilbert spaces.

*(1)* $(\alpha T_1 + \beta T_2)^* = \bar{\alpha} T_1^* + \bar{\beta} T_2^*$,
*(2)* $(ST)^* = T^*S^*$ *where* $S \in B(X_2, X_3)$ *and* $X_3$ *is a Hilbert space,*
*(3)* $(T^{-1})^* = (T^*)^{-1}$ *if* $T \in B(X_1)$ *is invertible.*

Consider $T \in L(\mathbb{F}^n, \mathbb{F}^n)$ where $\{e1, \cdots, en\}$ and $\{f1, \cdots, fm\}$ denote the canonical bases in $\mathbb{F}^n$ and $\mathbb{F}^m$ respectively. Then $Te_j = \sum_1^m a_{k_j} f_k$ where $(a_{k_j})$ is the matrix associated with $T$, and $T^*f_k = \sum_1^n b_{j_k} e_j$ where $(b_{j_k})$ is the matrix associated with $T^*$. Letting $x = \sum_1^n \alpha_j e_j$ and $y = \sum_1^m \beta_k f_k$ , then

$$\langle Tx, y\rangle = \langle \sum \alpha_j Te_j, \sum \beta_k f_k\rangle = \sum \alpha_j \overline{\beta_k} a_{kj},$$
$$\langle x, T^*y\rangle = \langle \sum \alpha_j e_j, \sum \beta_k T^* f_k\rangle = \sum \alpha_j \overline{\beta_k b_{jk}},$$

From $\langle Tx, y\rangle = \langle x, T^*y\rangle =$ we conclude $(bij) = (aji)$.

So the matrix associated with $T*$ is the adjoint matrix of the matrix associated with $T$. This justifies the terminology of the adjoint of a linear operator. Let $X$ be a Hilbert space. A bounded linear operator on $X$ to itself is called **self-adjoint** if $T^* = T$. For $T \in B(\mathbb{F}^n)$ its associated matrix satisfies $(ajk) = (akj)$. That is to say, it is a self-adjoint matrix.
When the scalar field is real, the matrix is called symmetric. In some

texts, the terminology a "symmetric operator" is used instead of a "self-adjoint operator", and a self-adjoint operator is reserved for a densely defined unbounded operator whose adjoint is equal to itself. We never touch upon unbounded operators, so this definition will not come up; there is no chance to mess things up.

A basic property of a self-adjoint operator is that its eigenvalues must be real. Recall that $\lambda$ is an eigenvalue of a linear operator $T$ if there exists a non-zero vector $x$ satisfying $Tx = \lambda x$. The eigenspace $\Phi\lambda = \{x \in X : Tx = \lambda x\}$ forms a subspace of $X$ and it is closed when $T$ is bounded.

**Proposition9.3.3.** Let T ∈ B(X) be self-adjoint where X is a Hilbert space.

(1) All eigenvalues of T are real; and

(2) Eigenvectors corresponding to distinct eigenvalues are orthogonal.

*Proof.* (1). If $x$ is an eigenvector for the eigenvalue $\lambda$, then

$$\begin{aligned}\langle Tx, x\rangle &= \langle \lambda x, x\rangle = \lambda\langle x, x\rangle,\\ \langle x, Tx\rangle &= \langle x, \lambda x\rangle = \bar{\lambda}\langle x, x\rangle.\end{aligned}$$

By self-adjointness, $\lambda \langle x, x \rangle = \bar{\lambda} \langle x, x \rangle$ which implies $\lambda$ is real.

(2). Let $Tx_1 = \lambda_1 x_1$ and $Tx_2 = \lambda_2 x_2$, where $\lambda_1$ and $\lambda_2$ are distinct. We have

$$\begin{aligned}\langle Tx_1, x_2\rangle &= \lambda_1\langle x_1, x_2\rangle,\\ \langle x_1, Tx_2\rangle &= \lambda_2\langle x_1, x_2\rangle.\end{aligned}$$

By self-adjointness, $\lambda_1\langle x_1, x_2\rangle = \lambda_2\langle x_1, x_2\rangle$ and, as the eigenvalues are distinct, $\langle x_1, x_2\rangle = 0$

**Proposition 9.3 .4.** Let T ∈ B(X) be self-adjoint where X is a Hilbert space. Then

$||T|| = \sup\{|\langle Tx, x\rangle| : x \in X, ||x|| = 1.\}$

*Proof.* Denote the right hand side of the above formula by $M$. As

$|\langle Tx, x\rangle| \leq ||Tx||||x|| \leq ||T||||x||^2$,

taking supremum over all unit $x$ shows that $M \leq ||T||$.

On the other hand, from $\langle Tx, x\rangle = \langle x, Tx\rangle = \overline{\langle Tx, x\rangle}$ we know that $\langle Tx, x\rangle \in \mathbb{R}$, for all $x$. By a direct expansion

$$\begin{aligned}\langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle &= 2\langle Tx, y\rangle + 2\langle Ty, x\rangle \\ &= 4Re\langle Tx, y\rangle,\end{aligned}$$

because $T$ is self-adjoint. As a result,

$$\begin{aligned}Re\langle Tx, y\rangle &= \frac{1}{4}(\langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle) \\ &\leq \frac{M}{4}(\|x+y\|^2 + \|x-y\|^2) \\ &= \frac{M}{2}(\|x\|^2 + \|y\|^2),\end{aligned}$$

by the parallelogram identity. Taking $x$, $kxk = 1$, and $y = Tx/kTxk(6= 0)$

$$\|Tx\| = Re\langle Tx, y\rangle \leq \frac{M}{2}(1+1) = M,$$

whence $||T|| = M$.

**Remark** We shall use the following remarks in next section:

(a) sup$\{|\langle Tx, x\rangle = | : ||x|| = 1\}$ may be expressed as

(b) From this proposition, we know that $T \equiv 0$ if $\langle Tx, x\rangle = = 0$ for all $x$.

## 9.4 COMPACT, SELF-ADJOINT OPERATORS

A linear operator $T \in L(X1, X2)$ where $X1$ and $X2$ are normed spaces is called **compact** if whenever $\{x_n\}$, $||x_k|| \leq M$ for some $M$, $\{Tx_k\}$ has a convergent subsequence. In other words, the image of a bounded sequence under a compact operator has the Bolzano-Weierstrass property. A compact operator is necessarily bounded. It is like a "regulator" which produces finite dimensional behavior. All compact operators form a closed subspace of $B(X)$ where $X$ is a Banach space under the operator norm. Furthermore, it is a two-sided ideal in the sense that $ST$ and $TS$ are compact if $T$ is compact and $S$ is bounded. The transpose (or the adjoint when the space is Hilbert) of a compact operator is again compact. It is a good exercise to prove all these facts. A common class of compact operators is provided by integral operators. Letting $K \in C([a, b]2)$ and considering the operator

$$\mathcal{I}f(x) = \int_a^b K(x, y)f(y)dy,$$

we saw in Chapter 4 that $I$ is bounded on $C[a, b]$ as well as $L^p((a, b))$, $p \in [1, \infty)$. (Forgive me for abusing the same notation.) We claim that it is compact on any one of these spaces. Let's take it to be $L^p((a, b))$, $p \geq 1$. Let $\{fj\}$ be a sequence in $L^p((a, b))$, $||f_j||_p \leq M$, say. By the definition of $L^p$-space, we can find $g_j \in C([a, b])$ such that $||fj - gj||_p < 1/j$. Then

$$\mathcal{I}g_j(x) = \int_a^b K(x, y)g_j(y)dy$$

makes sense. As $K$ is uniformly continuous, for any $\varepsilon > 0$ there exists $\delta$ such that $|K(x, y) - K(x', y')| < \varepsilon$ for

$$\sqrt{(x - x')^2 + (y - y')^2} < \delta.$$

So,

$$\begin{aligned} |\mathcal{I}g_j(x) - \mathcal{I}g_j(x')| &\leq \int_a^b |K(x, y) - K(x', y)||g_j(y)|dy \\ &\leq \varepsilon(b - a)^{1/q}\|g_j\|_p \\ &\leq \varepsilon(b - a)^{1/q}(1 + M), \end{aligned}$$

where $q$ is conjugate to $p$. We conclude that $\{I_{g_j}\}$ is equicontinuous in $[a, b]$. Similarly we can show that it is also uniformly bounded. Hence by Arzela-Ascoli theorem there exists $\{I_{g_{j_k}}\}$ converging uniformly to some

$h \in C[a, b]$. As uniform convergence is stronger than $L^p$-convergence, we have

$$\|\mathcal{I}f_{j_k} - h\|_p \leq \|\mathcal{I}f_{j_k} - \mathcal{I}g_{j_k}\|_p + \|\mathcal{I}g_{j_k} - h\|_p \to 0,$$

as $k \to \infty$, so $I$ is compact.

Another subclass of compact operators is provided by operators of finite rank. A bounded linear operator $T$ is an **operator of finite rank** if its image is a finite dimensional subspace. Since $\{T_{x_k}\}$ is a bounded subset in a finite dimensional space whenever $\{x_k\}$ is bounded, clearly the Bolzano-Weierstrass property holds for it. In practise most compact operators are limits of operators of finite rank. For operators on Hilbert spaces, this can be established without much difficulty. For many years it was conjectured that this be true on Banach spaces, but now people have found sophisticated counterexamples even in a separable Banach space.

Here we consider linear operators in a Hilbert space to itself which is self-adjoint and compact simultaneously. The study of self-adjoint, compact operators was due to Hilbert and is an early success of functional analysis. There is a lot of information one can retrieve.

**Proposition 9.4.1.** Let T be compact, self-adjoint in B(X) where X is a Hilbert space X. Then

(1) For any non-zero eigenvalue λ, the eigenspace of λ, Φλ, is a finite dimensional subspace.

(2) If eigenvalues $\{\lambda_k\}$, where all $\lambda_k$'s are all distinct, converges to $\lambda*$, then $\lambda* = 0$.

*Proof.* The following proof works for (1) and (2). Assume on the contrary that there are infinitely many distinct eigenvectors. Let $\lambda j$ be a sequence of eigenvalues of $T$, $\lambda_j \to \lambda^* \neq 0$ and $Tx_j = \lambda_j x_j$, $||x_j|| = 1$ where $\{x_j\}$ forms an orthonormal set. According to Proposition 9.2.3 (b), $||x_j - x_k|| = \sqrt{2}$. On the other hand, by compactness there exists $T_{x_{j_k}} \to x_0$. That is to say, $\lambda j||x_j|| \to x_0$. By assumption, $\lambda jk \to \lambda^*$.

It follows that

$$x_{j_k} = \frac{1}{\lambda_{j_k}} \lambda_{j_k} x_{j_k} \to x_0/\lambda^*.$$

So $\{x_{j_k}\}$ is a Cauchy sequence and $||x_{j_k} - x_{j_l}|| \to 0$ as $jk \neq jl \to \infty$, contradicting $||x_{j_k} - x_{j_l}|| = \sqrt{2}$.

**Lemma 9.4.2.** *Let T be compact, self-adjoint in* $B(X)$ *where X is a Hilbert space X. Then*

$$M = \sup_{x \neq 0} \frac{\langle Tx, x \rangle}{||x||^2},$$

is an eigenvalue of T provided it is positive. Similarly,

$$m = \inf_{x \neq 0} \frac{\langle Tx, x \rangle}{||x||^2}$$

is an eigenvalue of T provided it is negative.

*Proof.* It suffices to consider the first case. Let $\{x_k\}$ be a sequence satisfying $||x_k|| = 1$ and $\langle Tx_k, x_k \rangle \to M$. By compactness, there exists $Tx_{k_j} \to x_0$ in $X$.

Consider the self-adjoint operator $T - mI$. By Remark (a) after Proposition 9.2.4, $||T - mI|| = \max\{M - m, m - m\} = M - m$. We have

$$\begin{aligned}
\|Tx_k - Mx_k\|^2 &= \|(T - mI)x_k - (M - m)x_k\|^2 \\
&= \|(T - mI)x_k\|^2 + (M - m)^2\|x_k\|^2 - \langle (T - mI)x_k, (M - m)x_k \rangle \\
&\quad - \langle (M - m)x_k, (T - mI)x_k \rangle \\
&\leq \|T - mI\|^2 + (M - m)^2 - 2(M - m)(\langle Tx_k, x_k \rangle - m) \\
&= 2(M - m)^2 - 2(M - m)(\langle Tx_k, x_k \rangle - m) \\
&\to 0
\end{aligned}$$

as $k \longrightarrow \infty$. Restricting to the subsequence $\{x_{k_j}\}$ to get $Txk$

$$Tx_{k_j} - Mx_{k_j} \to 0$$

As $Tx_{k_j} \longrightarrow x_0$, $x_{k_j} \longrightarrow x_0/M$, $x_0 \neq 0$, and so by continuity $Tx_{k_j} \longrightarrow Tx_0/M$.

We conclude that $x_0$ is an eigenvector to to the eigenvalue $M$.

The following is the main result of this chapter. It is an infinite dimensional version of the reduction to principal axes for a self-adjoint matrix. It is also called the spectral theorem for compact, self-adjoint operators.

**Theorem 9.4.3.** *Let T be compact, self-adjoint in B(X) where X is Hilbert space.*

(1) Suppose $\langle Tx, x \rangle > 0$ for some $x \in X$. Then

$$\lambda_1 = \sup_{x \neq 0} \frac{\langle Tx, x\rangle}{\|x\|^2}$$

is a positive eigenvalue of T.

(2) Recursively define, for $n \geq 2$

$$\lambda_n = \sup\{\frac{\langle Tx, x\rangle}{\|x\|^2} : x \neq 0, x \perp \langle x_1, \cdots, x_{n-1}\rangle\}$$

where $x_j$ satisfies $Tx_j = \lambda_j x_j$, $\|x_j\| = 1$, $j = 1, 2, \cdots, n-1$. Then $\lambda_n$ is a positive eigenvalue of T as long as the supremum is positive. The collection is finite when there exists some N such that

$\langle Tx, x\rangle \leq 0$, for all $x \perp \langle x_1, \cdots, x_N\rangle$ .

Otherwise, there are infinitely many λj's and $\lambda_1 \geq \lambda_2 \geq \cdots \to 0$.

(3) For any "eigenpair" (λ, z) where λ > 0, λ must equal to λj for some j and z belong to the subspace spanned by all xj. (We note that to the same λj there could be more than one corresponding eigenvectors by the above construction.)

(4) Similarly, all negative eigenvalues are given by

$$\lambda_1' = \inf_{x \neq 0} \frac{\langle Tx, x \rangle}{\|x\|^2},$$

*and, for* $n \geq 2$,

$$\lambda_n' = \inf\{\frac{\langle Tx, x \rangle}{\|x\|^2} : x \neq 0, x \perp \langle x_1', \cdots, x_{n-1}' \rangle\}$$

if $\langle Tx, x \rangle < 0$ for some x. Here $x_j'$ is the normalized eigenvector of $\lambda j'$.

(5) Let $\langle x_k, x_k' \rangle$ be the span of all normalized eigenvectors. Then

$$X = \overline{\langle x_k, x_k' \rangle} \oplus X_0,$$

where X' is the zero-eigenspace of T .

*Proof.* (1) follows directly from Lemma 9.3.2.

(2) Consider the closed subspace $X_1 = \langle x_1 \rangle^{\perp}$. We check that $T : X_1 \mapsto X_1$. For, if $x \perp x_1$, then $0 = \langle Tx, x_1 \rangle = \langle x, Tx_1 \rangle = \lambda_1 \langle x, x_1 \rangle$, so $Tx \perp x_1$. It is routine to check that $T : X1 \mapsto X1$ is still a compact, self-adjoint operator. By applying Lemma 9.3.2 again

$$\lambda_2 = \sup\{\frac{\langle Tx, x \rangle}{\|x\|^2} : x \neq 0, x \in X_1\}$$

is an eigenvalue provided the supremum is positive. We may repeat this process to obtain the other eigenvalues until there exists an $N$ such that $\langle Tx, x_1 \rangle \leq 0$, for all $x \perp \langle x_1, \cdots, x_N \rangle$. Otherwise, we have an infinite sequence of decreasing eigenvalues. By Proposition 6.5, this sequence must converge to zero.

(3) Suppose $\lambda$ is a positive eigenvalue with eigenvector $\tilde{x}$. If $(\lambda, \tilde{x}.)$ does not come from the above construction, we must have $\lambda \leq \lambda 1$ and there



$$\lambda = \frac{\langle T\tilde{x}, \tilde{x} \rangle}{\|\tilde{x}\|^2} \leq \sup\{\frac{\langle Tx, x \rangle}{\|x\|^2} : x \neq 0, x \perp \langle x_1, \cdots, x_n \rangle\} = \lambda_{n+1},$$

exists some $n$ such that $\lambda \in (\lambda_{n+1}, \lambda_n]$ or $(0, \lambda_N]$ (when $N < \infty$). Consider the former first. When $\lambda$ is not equal to $\lambda_n$, $x$ e is orthogonal to all $xn$. However, by the construction of $\lambda_{n+1}$, we have

contradiction holds. When $\lambda = \lambda_n$, let us assume $\lambda_{n-K} > \lambda_{n-K+1} = \lambda_{n-K+2} = \cdots = \lambda n > \lambda_{n+1}$ because of finite multiplicity. The modified vector $\bar{x} \equiv \tilde{x}$ e $- P\tilde{x}$ where $P$ is the orthogonal projection of $\tilde{x}$ to the subspace spanned by $\{x_{n-K+1}, \cdots, x_n\}$ is orthogonal to all $\{x1, x2, \cdots, x_n\}$ and still satisfies $Tx = \lambda x$. Without loss of generality we may assume $x = \tilde{x}$. Then the above argument still produces a contradiction. A similar argument works for $\lambda \in (0, \lambda_N]$.

(4) The proof is left to the reader as an exercise.

(5) By the above construction we see that for all $x$ in $Z \equiv \overline{\langle x_k, x'_k \rangle}^{\perp}$, $\langle Tx, x \rangle = 0$. It is readily checked that $T$ maps $Z$ to itself. By Remark (b) after Proposition 6.4 we conclude that so $T \equiv 0$ on $Z$. In other words, $\overline{\langle x_k, x'_k \rangle}^{\perp}$, is the 0-eigenspace. By the theorem on orthogonal decomposition

$$X = \overline{\langle x_k, x'_k \rangle} \oplus \overline{\langle x_k, x'_k \rangle}^{\perp} = \overline{\langle x_k, x'_k \rangle} \oplus X_0.$$

This theorem may be viewed as the statement: Any matrix representation of a compact, self-adjoint operator can be diagonalized by a “rotation”. Let us take the field to be real and consider $T$ a symmetric linear transformation on the Euclidean space $\mathbb{R}^n$. For any orthonormal basis $\{x1, \cdots, xn\}$, the matrix $A \equiv (a_{jk})$, $Tx_k = \sum_j a_{jk} x_j$, is the matrix representation of $T$ with respect to this basis. From linear algebra we know there exists an orthogonal matrix $R$ such that $R^* AR$ is equal to a diagonal matrix $\Lambda$. (An orthogonal matrix $R$ satisfies $R^*R = I$ by definition.) Letting $y = Rx$, the matrix representation of $T$ with respect to the new orthonormal basis $\{y1, \cdots, yn\}$ is the matrix $\Lambda \equiv (\lambda_j \delta_{jk})$ where $y_j$ is the eigenvector of the eigenvalue $\lambda_j$.

Now, for a compact, symmetric operator on an infinite dimensional Hilbert space the same thing happens. Let us assume for simplicity that zero is not an eigenvalue. Then for any complete orthonormal set $\{x_j\}$ the operator $T$ is represented by an infinite matrix $A \equiv (a_{jk}), j, k \geq 1$, defined similarly as above. Let $\{\lambda 1, \lambda 2, \ldots, \}$ be an ordering of all eigenvalues of $T$ according to $|\lambda 1| \geq |\lambda 2| \geq |\lambda 3| \geq \ldots$ and $zj$ the corresponding (orthonormal) eigenvectors. According to the theorem, $\{zj\}$ forms a complete orthonormal set and the mapping defined by $zj = \sum_k r_{jk} x_k$ defines an "orthogonal matrix" $R \equiv (rjk)$ which satisfies $R^*R = I$. We have $R{*}AR = \Lambda$ where $\Lambda$ is the diagonal matrix consisting of eigenvalues.

Consider the set $\{x : \sum a_{jk} x_j\, x_k = 1\}$ where $(a_{jk})$ is positive definite. The discussion above shows that in the new coordinates given by $yj$'s, as a result of a rotation of $xj$'s, this set becomes $\{y : \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2 = 1, \}$, an ellipsoid in principal axes. In the infinite dimensional setting one may still call the eigenvectors $zj$ the principal axes of $T$ , and the theorem guarantees such reduction to principal axes by a rotation is always possible.

**CHECK YOUR PROGRESS**

1. Explain : A Hilbert space has a countable complete orthonormal set if and only if it is separable in its induced metric

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2. Explain Adjoint operator

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3. What do you understand by compact operator

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## 9.5 LET'S SUM UP

We have studied the spectrum theory and its application. We understood the cocept of ajoint and compact operator and their relation.

## 9.6 KEYWORDS

**Subsequence** – a **subsequence** is a sequence that can be derived from another sequence by deleting some or no elements without changing the order of the remaining elements.

**Closed subspace** - A **closed subspace** is a **subspace** that when treated as a **subset** of the original space is a **closed** set in the original topology.

**Two sided ideal** - The term **two-sided** ideal is used in noncommutative rings to denote a subset that is both a right ideal and a left ideal. In commutative rings, where right and left are equivalent, a **two-sided** ideal is simply called "the" ideal

## 9.7 QUESTIONS FOR REVIEW

1. Let $T \in B(X)$. Show that Re $\langle Tx, x\rangle = 0$ implies $T + T^* = 0$

2. Under the identification of the dual space of a Hilbert space with itself by the Fr´echet-Riesz theorem, show that the transpose of $T \in B(Y', X')$ becomes the adjoint $T* \in B(Y, X)$. Note: The identification is sesquilinear.

3. Let $T \in L(X)$ be a compact operator. Show that
(a) $T$ is bounded,
(b) for any $S \in B(X)$, $TS$ and $ST$ are compact operators; and
(c) all compact operators form a closed set in $B(X)$. Hint: Use a diagonal sequence.

## 9.8 SUGGESTED READINGS

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## 9.9 ANSWER TO CHECK YOUR PROGRESS

1. Provide proof – 9.2.1

2. Provide explanation–9.3

3. Provide explanation–9.4

# UNIT 10: WEAK COMPACTNESS

**STRUCTURE**



## 10.0 OBJECTIVE

Understand the concept of Weak Sequential Compactness

Enumerate Topologies Induced by Functionals

Comprehend Weak and Weak* Topologies

## 10.1 INTRODUCTION

An essential difference between finite and infinite dimensional normed spaces is that the closed unit ball is compact in the former but not compact in the latter. To compensate the loss of compactness in an infinite dimensional space, one may impose additional conditions to sustain compactness. A complete answer is known for the space of continuous functions under the sup-norm, see the discussion on AscoliArzela theorem. Yet there is a more radical way of thinking, namely, we search for a weaker concept of compactness which an infinite dimensional closed unit ball satisfies. This leads us to the study of both weakly sequential and weak compactness. A weak topology contains less open sets than the topology induced by the norm (the strong topology), so the chance of obtaining compact sets is higher.

In the first section of this chapter, we discuss weak sequential convergence and prove the widely used result that the closed unit ball is weakly sequentially compact in a reflexive space. To study the problem in a general normed space, new concepts of weak and weak* topologies are introduced.

In Section 2 we discuss some basic properties of the topologies induced by a family of linear functionals on a vector space. By specifying these families to *X0* on *X* and *X* on *X0* (through the canonical identification), we obtain the weak and weak* topologies on the spaces *X* and *X0* respectively. In Section 3 we prove the central results in this chapter, namely, Alaoglu theorem and a theorem characterizing reflexive spaces by weak compactness. We conclude this chapter by a discussion on extreme points in a convex set and a proof of Krein-Milman theorem, a cornerstone in functional analysis. For simplicity we will take the scalar field to be real. The reader should have no difficulty in extending the results to the complex field.

## 10.2 WEAK SEQUENTIAL COMPACTNESS

Let $(X, ||\cdot||)$ be a normed space and $X'$its dual. A sequence *{xn}* in $X$ is called **weakly convergent** to some $x \in X$ if for every $\Lambda \in X'$,

$$\Lambda x_n \to \Lambda x, \qquad \text{as } n \to \infty.$$

Denote it by $x_n \rightharpoonup x$.

We will call the convergence of a sequence "strong convergence" in contrast to weak convergence. The following proposition clarifies the relationship between these two notions of convergence.

**Proposition 10.2.1.** *Let* $(X, ||\cdot||)$ *be a normed space and* $\{x^n\} \subset X$.

*(a)* $x_n * x$ *and* $xn * y$ *implies* $x = y$.

*(b)* $x_n \to x$ *implies that* $x_n * x$.

(c) If xn * x, then ‖xn‖ ≤ C, ∀n for some constant C.

(d) If $x_n$ * x, then ‖x‖ ≤ $\lim_{n\to\infty}$ ‖ $x_n$ ‖.

(e) If $x_n$ * x, then x belongs to the closure of the convex hull of { $x_n$ }.

Note that (d) can be deduced from (e) in this proposition. However, a short, direct proof is preferred.

*Proof.* (a) From $x_n * x$ and $x_n * y$ we deduce that $\Lambda(x - y) = 0$ for all $\Lambda \in X'$ and $x = y$.

(b) { $x_n$ } converges to $x$ strongly means that $||xn – x|| \to 0$ as $n \to \infty$. For $\Lambda \in X'$,

$$|\Lambda x_n - \Lambda_x| \leq ||\Lambda||||xn - x|| \to 0,$$

so $\{x_n\}$ converges to $x$ weakly.

(c) Since for each $\Lambda \in X'$, $\Lambda xn \to \Lambda x$, so $\Lambda xn$ is bounded. The conclusion follows immediately from the uniform boundedness principle. You should note that $X'$ is a Banach space.

(d) Pick $\Lambda_1 \in X'$ satisfying $\Lambda_1 x = ||x||$ and $||\Lambda 1|| = 1$, that is, it is a dual point of $x$. For any
convergent subsequence of $\{||xn||\}$, $\{||x_{n_j}||\}$, we have

$$\begin{aligned}
\|x\| &= |\Lambda_1 x| \\
&\leq |\Lambda_1(x - x_{n_j})| + |\Lambda_1 x_{n_j}| \\
&\leq |\Lambda_1(x - x_{n_j})| + \|x_{n_j}\| \\
&\to \lim_{j\to\infty} \|x_{n_j}\|,
\end{aligned}$$

whence (d) follows.

(e) Let $K$ be the closure of the convex hull of $\{xn\}$. If, on the contrary, $x$ does not belong to $K$, by
the separation form of the Hahn-Banach theorem, there exist some $\Lambda \in X'$ and $\alpha$ such that

$$\Lambda x < \alpha < \Lambda y, \ \forall y \in K.$$

In particular, taking $y = x_n$ and letting $n \to \infty$, we have

$$\Lambda x < \alpha \leq \lim_{n\to\infty} \Lambda x_n = \Lambda x,$$

Contradiction holds.

Proposition 10.1.1(b) shows that strong convergence implies weak convergence. When $X$ is of finite dimension, every element is of the form $x = \sum \alpha_j z_j$ after a basis $\{z1, \cdots, zn\}$ has been chosen. Consider the $n$ many linear functionals given by $\Lambda j(x) = \alpha j, j = 1, \cdots, n$. When $xk * x$ where $xk = \sum \alpha_j^k z_j$ and $x = \sum \alpha_j z_j$ , we have $\Lambda j(xk) = \alpha j\ k \rightarrow \Lambda j(x) = \alpha j$. It shows that $xk \rightarrow x$, that is, weak convergence also implies strong convergence. So they are equivalent when the space is finite dimensional. However, for infinite dimensional spaces this is rare. There are plenty weakly sequentially convergent sequences which are not strongly convergent. Let us look at two examples.

**Example** Consider $\ell^p$-space, $1 < p < \infty$ and $\{ej\} \subset \ell^p$ where $ej$'s are the "canonical vectors". It is clear that $\{ej\}$ does not have any convergent subsequence as $kei - ejkp = 21/p$ for distinct $i$ and $j$. On the other hand, we claim that $ej * 0$. To see this, recall that any bounded linear functional $\Lambda$ on \`$p$ can be identified with

$$\Lambda x = \sum_{j=1}^{\infty} y_j x_j, \quad x = (x_1, x_2, \cdots, x_n, \cdots),$$

where $y = (y_1, y_2, \cdots, y_n, \cdots)$, $\sum |y_j|^q < \infty$, by \`$p$-\`$q$ duality. We have $|\Lambda e_j| = |y_j|$. As $\sum |y_j|^q < \infty$, $|y_j| \rightarrow 0$, that is, $e_j * 0$ as $j \rightarrow \infty$.

**Example** Consider $\{f_n\}$, $f_n(x) = \sin nx$, in $L^2(0, 1)$. By a direct calculation, we have

$$\int_0^1 |f_n - f_m|^2 = 1 + O(\frac{1}{n}) + O(\frac{1}{m}), \quad \text{as } n, m \to \infty,$$

which means that this sequence does not converge in $L^2(0, 1)$. Nevertheless, let us show that it is weakly convergent to zero. First, we claim that

$$\int_0^1 x^m \sin nx dx \to 0, \qquad \text{as } n \to \infty,$$

for every monomial $xm$. Indeed, this follows easily from the formulas and

$$\int_0^1 x^m \sin nx dx = -\frac{\cos n}{n} + \frac{m}{n}\int_0^1 x^{m-1}\cos nx dx,$$

$$\int_0^1 \sin nx dx = \frac{1-\cos n}{n}.$$

$$\int_0^1 p(x)\sin nx dx \to 0, \quad \text{as } n \to \infty,$$

Cosequently, for every polynomial $p$. As all polynomials form a dense set in $L^2(0, 1)$, a density argument shows that the above formula also holds when $p$ is replaced by an $L^2$-function. By self-duality, we conclude that $\{\sin nx\}$ converges to 0 weakly in $L^2(0, 1)$.

Later, we will see that in an infinite dimensional reflexive (Banach) space, divergent sequences which are weakly convergent always exist. However, in some exceptional cases things may behave differently. A result of Schur asserts that any weakly convergent sequence in $\ell^1$ is also strongly convergent, see exercise.

The most important and useful result concerning weak sequential compactness is the following theorem.

**Theorem 10.2.2.** Every closed ball in a reflexive space is weakly sequentially compact.

A set $E$ in $(X, ||\cdot||)$ is **weakly sequentially compact** if every sequence in it contains a weakly convergent subsequence in $E$.

***Proof.*** Without loss of generality we assume the ball is given by $\{x \in X : ||x|| \leq 1\}$. Let $\{x_n\}$ be a sequence contained in this ball. We would like to extract a weakly convergent subsequence from it.

Let $Y = \langle x_n \rangle$ be the closed subspace of $X$ spanned by $\{x_n\}$. It is clear that $Y$ is separable. As any closed subspace of a reflexive space is reflexive, $Y$ is also reflexive. Recalling that a normed space is separable when its dual is separable, we conclude from the relation $(Y')' = Y$ and the separability of $Y$ that $Y'$ is also separable. Let $S$ be a countable dense set in $Y'$. By extracting a diagonal sequence, we find a subsequence of $\{x_n\}$, $\{y_n\}$, such that

$$\lim_{n\to\infty} \Lambda y_n \text{ exists for every } \Lambda \in S. \quad (1)$$

For any $\Lambda \in Y'$, we can pick a sequence $\{\Lambda j\}$ from $S$ such that $||\Lambda - \Lambda j|| \to 0$ as $j \to \infty$. We claim that $\{\Lambda y_n\}$ is a Cauchy sequence in R. For, taking any $\varepsilon > 0$, we fix $j_0$ such that $||\Lambda - \Lambda j_0|| < \varepsilon$. Then

$$\begin{aligned} |\Lambda y_n - \Lambda y_m| &\leq |(\Lambda - \Lambda_{j_0})y_n| + |\Lambda_{j_0} y_n - \Lambda_{j_0} y_m| + |(\Lambda_{j_0} - \Lambda)y_m| \\ &< 2\varepsilon + |\Lambda_{j_0} y_n - \Lambda_{j_0} y_m|. \end{aligned}$$

By (1), there exists $n_0$ such that $|\Lambda j_0 y_n - \Lambda j_0 y_m| < \varepsilon$, for all $n, m \geq n_0$, so

$$|\Lambda y_n - \Lambda y_m| < 3\varepsilon,$$

that is, $\{\Lambda y_n\}$ is a Cauchy sequence. Define a real-valued function  on $Y'$ by

$$\ell(\Lambda) = \lim_{n\to\infty} \Lambda y_n.$$

It is readily checked that `(Λ) is linear. Moreover, we have

$$\begin{aligned} |\ell\Lambda| &= \lim_{n\to\infty} |\Lambda y_n| \\ &\leq ||\Lambda|| \overline{\lim_{n\to\infty}} ||y_n|| \\ &\leq ||\Lambda||, \end{aligned}$$

which means that $\ell \in Y''$.

By the reflexivity of $Y$, there exists some $y \in Y$ such that $\Lambda y = \ell(\Lambda)$. We conclude that $\Lambda y_n \to \Lambda y$ for every $\Lambda \in Y'$. Since each $\Lambda \in X'$ is a bounded functional on $Y$ by restriction, $y_n * y$. By Proposition 10.1.1 (c), $||y|| \leq 1$. The proof is completed.

**Corollary 10.2.3.** Let C be a non-empty convex set in a reflexive space X. It is weakly sequentially compact if and only if it is closed and bounded.

Proof. As C is bounded, it is contained in some closed ball B. By Theorem 10.1.2, any sequence $\{x_n\}$ in C has a subsequence {xni} weakly converging to some $x \in B$. As C is closed and convex, $x \in C$ according to Proposition 7.1(d), so C is weakly sequentially compact.

Conversely, let $\{x_n\}$ be a sequence in *C* which converges to some *x* in *X*. We would like to show that *x* belongs to *C*, so that *C* is closed. In fact, as *C* is weakly sequentially compact, there exists a subsequence $\{xn_j\}$ of $\{x_n\}$ which converges weakly to some *y* in *C*. By the uniqueness of limit, we conclude that *x* is equal to *y*, so it belongs to *C*. On the other hand, if there is some $\{xn\} \in C$, $||x_n|| \to \infty$, by weak sequential compactness there exists a weakly convergent subsequence *xnj* . However, by Proposition 10.1.1 (c), this subsequence is bounded, contradiction holds. Hence *C* must be bounded.

Recall that in a finite dimensional normed space a set is sequentially compact if and only if it is closed and bounded. We have generalized it to convex sets in a reflexive space simply by replacing sequential compactness by weak sequential compactness.

Parallel to weak sequential convergence, we call a sequence $\{\Lambda k\}$ in the dual space *X'* **weakly* sequentially convergent** to some $\Lambda$ if $\Lambda_k x \to \Lambda x$ for every $x \in X$. Weak* sequential compactness for a set in *X′* can be defined correspondingly. We refer to the exercises for some properties of this notion.

We conclude this section with an application of weak sequential compactness. More applications can be found in exercises. We examine again the problem of best approximation. In Theorem 5.8 we showed that this problem always admits a unique solution in a Hilbert space. Now, we have

**Theorem 10.2.4.** Let X be a reflexive space and C a nonempty closed, convex subset. Then for any $x \in X$, there exists $z \in C$ such that

$$||x - z|| = \inf\{||x - y|| : y \in C\}.$$

In other words, the best approximation problem always has a solution in a reflexive space.

***Proof.*** Let $\{y_n\}$ be a minimizing sequence of the problem, that is,

$$\|x - y_n\| \to d \equiv \inf\{\|x - y\| : \quad y \in C\}.$$

From

$$\|y_n\| \leq \|x - y_n\| + \|x\| \to d + \|x\|,$$

we see that $\{y_n\}$ is a bounded sequence in $X$. By Theorem 7.2, it contains a weakly convergent subsequence $\{y_{n_j}\}$, $y_{n_j}$ * $z$ for some $z$.

Proposition10.1.1.(d) asserts that $z \in C$. Moreover, by the same proposition we have so $z$ is a point in $C$ realizing the distance.

$$\|x - z\| \leq \varliminf_{j \to \infty} \|x - y_{n_j}\| = d,$$

## 10.3 TOPOLOGIES INDUCED BY FUNCTIONALS

In this section we will first give a quick review on some basic topological concepts, especially those concerning the topology induced by a family of functions on a set. Next, we examine more closely about the case where the set is a vector space and the functions are linear functionals on this vector space.

Recall that $(X, \tau)$ where $X$ is a set and $\tau$ is a collection of subsets of $X$ is called a topological space if $\tau$ satisfies

(a) The empty set $\emptyset$ and $X$ belong to $\tau$,

(b) unions of elements in $\tau$ belongs to $\tau$, and

(c) intersections of finitely many elements in $\tau$ belongs to $\tau$.

Any element in $\tau$ is called an open set. A set $F$ is closed if its complement is open. Immediately we
deduce from (a), (b), and (c) the following facts:

(d) $X$ and $\emptyset$ are closed sets,

(e) intersections of closed sets are closed sets, and

(f) unions of finitely many closed sets are closed sets.

For any subset $E$ of $X$, its closure is defined to be

$$\overline{E} \equiv \bigcap \{F : \ F \text{ is a closed set containing } E\}.$$

Note that $X$ is closed and it contains $E$. By (e) $\overline{E}$ is a closed set. Clearly, it is the smallest closed set containing $E$. A subset $K$ is compact if every open covering of $K$ has a finite subcover.

With open sets at hand, we can talk about convergence and continuity. For instance, a sequence $\{x_n\}$ in $X$ is convergent to some $x$ in $X$ if for each open set $G$ containing $x$, there exists some $n_0$ such that $x_n \in G$ for all $n \geq n_0$. A mapping $f : (X, \tau) \mapsto (Y, \sigma)$ between two topological spaces is **continuous** at $x$ if $f^{-1}(G)$ is open for any open set $G$ containing $f(x)$. It is continuous in a subset $E$ if it is continuous at every $x$ in $E$.

In a metric space $(X, d)$, $G$ is an open set if for every $x \in G$, there exists some metric ball $B\rho(x) \subset G$. One can verify that the collection of all these open sets makes $X$ into a topological space. This is the topology induced by the metric $d$. The notions of open set, closed set, the closure of a set, convergence of a sequence and continuity of functions all coincide with those previously defined for a metric space.

However, caution must be made as many facts valid in a metric space are no longer true in a general topological space. For instance, a set in a metric space is closed if and only if the limit of any convergent sequence belongs to the set. In a general topological space, the "only if" part holds but the "if" part does not. Further, a set in a metric space is compact if and only if it is sequentially compact. This is not always true for a general topological space. There are compact topological spaces admitting sequences which do not have convergent sequences. On the other hand, there are non-compact topological spaces in which all sequences have convergent sequences. When it comes to continuity, a function $f$ is continuous at $x$ in a metric space if and only if for any sequence $\{ x_n \}$ converging to $x$, $f(x_n)$ converges to $f(x)$. In a topological space, convergence of $f(x_n)$ to $f(x)$ for any $\{ x_n \} \rightarrow x$ does not ensure

continuity, although it holds when *f* is continuous at *x*. In a word, topological properties cannot be fully described in terms of sequences in a general topological space.
Now, we turn to the topology induced by functions on a set.

Let *X* be a non-empty set. For a non-empty collection of functions *F* from *X* to R, we introduce a topology $\tau(X, F)$ on *X* by the following way. First, Let $\mathcal{U}_1$ be the collection of subsets of *X* of the form $f^{-1}(a, b)$ where $a, b \in$ R and *f* in *F*. (Define $\emptyset = f^{-1}(\emptyset)$.) Next, let $\mathcal{U}_2$ be the collection of all finite intersections of unions of elements from *U*1. Finally, let $\tau = \tau(X, F)$ contain all unions of elements from $\mathcal{U}_2$. One can verify that $\tau$ forms a topology on *X*. By this construction, each *f* in *F* is a continuous function in $(X, \tau)$. In fact, for any $(X, \tau 1)$ in which every function in *F* is continuous, $\tau 1$ must contain $\tau$.
In this sense $\tau$ is the weakest topology to make each element of $\mathcal{F}$ continuous. We call it the **induced topology** by $\mathcal{F}$. Intuitively speaking, the induced topology is finer (containing more open sets) if there are more functions in $\mathcal{F}$ and coarser (containing less open sets) if there are less functions in $\mathcal{F}$. A topological space $(X, \tau)$ is a Hausdorff space if for any two distinct points in *X*, there exist two disjoint open sets containing these points respectively. In analysis Hausdorff space is preferred for many of its nice properties. For instance, a compact set is closed in a Hausdorff space. A metric space is always Hausdorff, as any distinct *x*1 and *x*2 are contained in the disjoint open sets $\{z \in X : d(z, x_1) < 1/2d(x_1, x_2)\}$ and $\{z \in X : d(z, x_2) < 1/2d(x_1, x_2)\}$ respectively.
To make an induced topology a Hausdorff one, *F* cannot contain too few functions. It is called **separating** if for any two distinct points *x* and *y* in *X*, there exists a function $f \in F$ such that $f(x) \neq f(y)$.

**Proposition 10.3.1.** The space (X, τ(X, F)) is a Hausdorff space if F is separating.

***Proof***. For distinct *x*1 and *x*2, let $f \in F$ satisfy $f(x1) < \alpha < f(x2)$ for some $\alpha$, say. Then $G1 \equiv \{x : f(x) < \alpha\}$ and $G2 \equiv \{x : f(x) > \alpha\}$ are two disjoint open sets containing $x_1$ and $x_2$ respectively.

So far, $X$ has been taken to be a non-empty set without any extra structure and $F$ is a set of real functions on $X$. Now, let us assume that $X$ is a vector space and $F$ a subset of $L(X, \mathbb{R})$, that is, it is composed of linear functionals. We would like to know more about the induced topology in this setting.

**Proposition 10.3.2.** Consider the induced topology $\tau(X, F)$ where X is a vector space and $F \subset L(X, R)$. Let G be a non-empty set in X. We have

(a) G is open if and only if for each $x0 \in G$, there exists U of the form

$$U = \{x : |\Lambda jx| < \alpha, j = 1, \cdots, N\} \qquad (2)$$

*for some* $\Lambda j \in F$ *and* $\alpha > 0$ *such that* $U + x0 \subset G$.

*(b) G is open if and only if* $G + x0$ *is open for every* $x0 \in X$.

*(c) G is open if and only if* $\lambda G$ *is open for every* $\lambda \neq 0$.

From (b) and (c) we see that translations and multiplications by non-zero scalars are homeomorphisms with respect to $\tau(X, F)$.

***Proof.*** (a) Let $x0 \in G$. By the definition of $\tau(X, F)$ there exists a set of the form $V = \{x : \Lambda_j x \in (\alpha j, \beta j), j = 1, \cdots, N\}$ containing $x_0$ in $G$. It follows that $x - x0 \in \Lambda-1((\alpha j - \Lambda jx0, \beta j - \Lambda jx0))$ for $x \in V$. So, the set in (2) by taking $\alpha = \min j\{|\alpha j - \Lambda jx0|, |\beta j - \Lambda jx0|\}$ is an open set containing $x0$ in $G$. The converse is trivial from definition.

(b) For $x \in G$, there exists some $U$ as in (7.2) such that $U + x \subset G$. But then $U + (x + x0) \subset G + x0$.

(c) Argue as in (b).

It is convenient to call a set of the form (7.2) a "$\tau$-ball centered at 0" or simply a "$\tau$-ball". Unlike a metric ball, a $\tau$-ball is not only specified by its "radius" $\alpha$ (now a vector), but also the functionals $\Lambda j$'s. Under the induced topology $\tau(X, F)$, every element in $F$ is continuous. In a normed space we know that a linear functional is continuous if and only if it is bounded. Here we have a similar result.

**Proposition 10.3.3.** Let $\Lambda$ be a linear functional on $(X, \tau(X, F))$.

(a) $\Lambda$ is continuous if and only if it is continuous at one point.

(b) $\Lambda$ is continuous if and only if it is bounded on a $\tau$-ball.

***Proof***. (a) Assume that Λ is continuous at $x0$. For any ($\alpha, \beta$) containing Λ$x0$, Λ−1($\alpha, \beta$) has an open subset $U$ containing $x_0$. Let ($\alpha', \beta'$) be any open interval containing Λ$x$. Then ($\alpha, \beta$) = ($\alpha', \beta0$)+Λ$x0$−Λ$x$ is an open interval containing Λ$x0$, using Λ−1($\alpha, \beta$) = Λ−1($\alpha0, \beta0$) + $x0 - x$ and Proposition 10.2.2 (b), we see that $U - x0 + x$ is an open subset of Λ−1($\alpha0, \beta0$) containing $x$, so Λ is continuous at $x$.

(b) $\Lambda^{-1}(-1, 1)$ is open for a continuous Λ. As $0 \in \Lambda^{-1}(-1, 1)$, there is an open set of $U$ the form

(2) Contained in Λ−1 by Proposition 7.6 (a). So $|\Lambda(U)| \leq 1$, and Λ is bounded on $U$. Conversely, if $|\Lambda(U)| \leq M$ for some constant $M$ where $U$ is a $\tau$-ball. By (a) it suffices to show that Λ is continuous at 0, that is, $\Lambda^{-1}(a, b)$ is open for every $a, b, a < 0 < b$. Pick any $x_0 \in \Lambda-1(a, b)$, there is an $\varepsilon > 0$ such that $(\Lambda x_0 - \varepsilon, \Lambda x_0 + \varepsilon) \subset (a, b)$. Letting $V = 2M\,\varepsilon$ $U$, it is easy to see that $V + x_0$ is an open set containing $x_0$ and $V + x_0 \in$ Λ−1($a, b$), so Λ−1($a, b$) is open.

Under the topology $\tau(X, F)$, every element in $F$ is continuous by definition. Are there more? Consider the very special case where $F$ consists of a single functional Λ. Clearly, any constant multiple of Λ is continuous. Furthermore, one can show that the sum of two linear functionals from $F$ is continuous. The following proposition asserts that these are the only cases.

**Proposition 10.3.4.** Consider (X, τ(X, F)) where X is vector space and F ⊂ L(X, R). The collection of all continuous linear functionals is given by F if and only if F is a subspace of L(X, R).

***Proof.*** We will only prove the "if " part and leave the "only if " part as exercise.

Let Λ be continuous in $\tau(X, F)$. There exists an open set

$$U = \{x : \Lambda_j x \in (-\alpha, \alpha), j = 1, \cdots, N\} \subset \Lambda^{-1}(-1, 1).$$

We claim that $\Lambda$ vanishes on $\bigcap_{j=1}^{N} N(\Lambda j)$. For, if $z$ satisfies $\Lambda_j z = 0$, $j = 1, \cdots, N$, then $\Lambda_j(\lambda z) = 0$ for all $\lambda$, so $\lambda z \in U$. From

$$|\lambda||\Lambda z| = |\Lambda(\lambda z)| \leq 1$$

that $\Lambda z = 0$ after letting $|\lambda k$ go to infinity. By the lemma below, $\Lambda$ is a linear combination of $\Lambda j$, so $\Lambda \in F$ by assumption.

**Lemma 10.3.5.** *Let* $\Lambda, \Lambda_1, \cdots, \Lambda n$ *be in* $L(X, \mathrm{R})$. *If* $\Lambda x = 0$ *whenever* $x \in \bigcap_{j=1}^{N} N(\Lambda j)$, *then* $\Lambda$ *is a linear combination of* $\Lambda_1, \cdots,$ *and* $\Lambda n$.

*Proof.* Let $Z = \{(\Lambda x, \Lambda_1 x, \cdots, \Lambda_n x) : x \in X\}$. Clearly $Z$ is a subspace of $\mathrm{R}^{n+1}$ and it is proper because the point $(1, 0, \cdots, 0)$ does not belong to it by assumption. We can find a hyperplane $az + a_1 z_1 + \cdots + a_n z_n = 0$ which contains $Z$ but not $(1, 0, \cdots, 0)$. In other words,

$$a\Lambda x + a_1\Lambda_1 x + \cdots + a_n\Lambda_n x = 0,$$

for all $x \in X$ and $a1 + a10 + \cdots + an0 \neq 0$. The second expression shows that $a$ is non-zero, so the desired conclusion follows from the first expression.

We have discussed the separation theorem as a consequence of Hahn-Banach theorem. Now we establish a separation theorem in induced topology. It will be our main tool in later evelopment. We start with a lemma.

**Lemma 10.2.6.** *Let* $C$ *be an open, convex set in* $(X, \tau(X, F))$ *containing* 0 *and* $p$ *its gauge. Then*

$C = \{x : p(x) < 1\}$.

Recall that the gauge of a convex set is given by

$$p(x) = \inf\{\mu > 0 : \frac{1}{\mu} x \in C\},$$

and $p(x) = \infty$ if no such $\mu$ exists. It is a positive homogeneous subadditive function. When $C$ is open and contains 0, it contains some $\tau$-ball. Therefore, for every $x \in X$, we can find some small $\varepsilon > 0$ so that $\varepsilon x$ belongs to this $\tau$-ball and hence $C$, so $p(x)$ is always finite.

***Proof***. We claim that $\{p < 1\} \subset C$ for any convex set $C$ (not necessarily open) containing 0. Indeed, if $p(x) < 1$ for some $x$, then there exists some $\mu \in (0, 1)$ such that $\mu-1x \in C$. By convexity $x = (1 - \mu)0 + \mu(\mu-1x) \in C$.

To show the inclusion from the other direction, we observe for each $x$ in the open $C$, we can find a $\tau$-ball such that $U + x \subset C$. From the definition of $U$, there exists some small $\varepsilon > 0$ such that $\varepsilon x \in U$.

Thus, $x + \varepsilon x \in C$ and it implies that $p(x) \leq 1/(1 + \varepsilon) < 1$, the desired conclusion follows.

**Theorem 10.3.7.** Let A and B be two disjoint, non-empty convex sets in $(X, \tau(X, F))$ where X is a vector space and $F \subset L(X, R)$.

(a) When A is open, there exists a continuous linear functional $\Lambda$ such that

$$\Lambda x < \Lambda y, \qquad \text{for all } x \in A,\ y \in B.$$

(b) When A is compact and B is closed, there exist a continuous linear functional $\Lambda$, $\alpha$ and $\beta$ such

*that*

$$\Lambda x < \alpha < \beta < \Lambda y, \qquad \text{for all } x \in A,\ y \in B.$$

*Proof.* (a) Consider the convex set $C = A - B + x0$ where $x0$ is a point in $B - A$. It is open because $C = Sx{\in}B\ A - x + x0$ and $A$ is open. Moreover, it contains the origin as $x_0$ is located outside $C$. Let $p$ be the gauge of $C$. Define $\Lambda0$ on the one-dimensional subspace $\langle x_0 \rangle$by $\Lambda_0(\alpha x_0) = \alpha$. Then $\Lambda_0 \leq p$ on this subspace. This is trivial when $\alpha \leq 0$. When $\alpha > 0$, by Lemma 10.2.5 $p(\alpha x_0) = \alpha p(x_0) \geq \alpha$ as $x_0$ lies outside $C$. Appealing to the general Hahn-Banach theorem, we find an extension of $\Lambda_0$, $\Lambda \in L(X, R)$, satisfying $\Lambda \leq p$ in $X$. For, $x \in A$ and $y \in B$,

$$\Lambda(x - y + x_0) \leq p(x - y + x_0)$$

Holds. It follows that $\Lambda x < \Lambda y$ after using $\Lambda x_0 = 1$ and Lemma 10.2.5. We still have to show that $\Lambda$ is continuous. We pick a $\tau$-ball $U$ in $C$. Noting that $x \in U$ implies

$-x \in U$, we have $|\Lambda x| \leq p(x) < 1$ in $U$, so $\Lambda$ is continuous by Proposition 10.2.3(b).

(b) We use a compactness argument to show that there is an open set $V$ such that $A + V$ is disjoint from $B$. For each $x \in A$, as $X \setminus B$ is open, there exists $Vx = \{y : |\Lambda_j y| \leq 2\gamma_x, j = 1, \cdots, N\}, \gamma_x > 0$, so that $Ux \equiv Vx + x$ is disjoint from $B$. The collection of all open sets $1/2Vx + x$ forms an open cover of $A$. As $A$ is compact, there is a finite subcover given by, say, finitely many $1/2Vxk + xk$, $k = 1, \cdots, m$. Taking

$$V = \{y : |\Lambda jy| \leq \gamma\}, \text{ some } \gamma > 0,$$

where the $\Lambda\ j$'s are taken from all those linear functionals appearing in the definition of $Vxk$, one verifies that $A + V$ is an open, convex set disjoint from $B$.

By (a) there exists a continuous linear functional $\Lambda$ satisfying $\Lambda x < \Lambda y$ for all $x \in A + V$ and $y \in B$. It is elementary to show that a non-zero linear functional is an open map, so $\Lambda(A + V)$ is an open set in R. On the other hand, $\Lambda A$ is compact as the image of a compact set by a continuous functional. So (b) holds for some $\alpha$ and $\beta$.

**CHECK YOUR PROGRESS**

1. Explain: Let C be a non-empty convex set in a reflexive space X. It is weakly sequentially compact if and only if it is closed and bounded.

2. Explain Topologies induced by functionals

# 10.4 WEAK AND WEAK* TOPOLOGIES

Let $(X, ||\cdot||)$ be a normed space. The topology $\tau(X, X')$ is called the **weak topology** of $X$. This is the weakest topology to make every bounded linear functional continuous. As ensured by the HahnBanach theorem, there are sufficiently many elements in *X′* to separate points, the weak topology is Hausdorff. However, it contains much less open sets than the strong topology does when the space is infinite dimensional. In sharp contrast to norm topology, we have the following results.

**Proposition 10.4.1.** Let X be an infinite dimensional normed space. Every weakly open set contains an infinite dimensional subspace of X. A set is weakly open means that it is open in τ(X, X'). As a consequence, every non-empty weakly open set is unbounded in norm.

***Proof.*** As every weakly open set contains a weak ball (that's, $\tau(X, X0)$-ball) $U$, it suffices to prove the result for $U$. Consider the linear map from $X$ to R$N$ given by $\Phi(x) = (\Lambda 1x, \cdots, \Lambda Nx)$ where $\Lambda j$'s are the bounded linear functionals defining $U$. The kernel of Φ is of infinite dimension. For any $x \in N(\Phi)$, $\Lambda_j x = 0$ for all $j$, so $N(\Phi) \subset U$.

A topological space is called **metrizable** if its topology is induced by some metric.

**Proposition 10.4.2** The weak topology is not metrizable when X is an infinite dimensional normed space.

***Proof.*** Assume the weak topology on $X$ comes from a metric $d$. As the topology induced from a metric admits a countable local base given by $\{x : d(x, x0) < 1/n\}$, $n \geq 1$, at every point $x0$, in particular, there is a countable base at 0 consisting of weak balls $Un = \{x : |\Lambda_j^n x| < \alpha n\}$, where $j = 1, 2, \cdots, N(n)$, $n \geq 1$.

All these $\Lambda_j^n$ 's form a countable set in $X0$. As $X0$ is a Banach space and every Hamel basis of a Banach space is uncountable (Proposition 4.14), we can find some $T \in X0$ which is independent of all these $\Lambda_j^n$ 's.

Consider the open set $G$ given by $\{x : |Tx| < 1\}$. It must contain some $Un0$, so $T$ vanishes on $\bigcap_j N(\Lambda_j^{n_0})$. However, by Lemma 10.2.4, $T$ is a linear combination of $\Lambda_j^{n_0}$'s, contradiction holds. Hence the weak topology is not metrizable.

Although the weak and norm topologies are very different as seen from the above propositions, they have something in common.

**Proposition 10.4.3.** A convex set in a normed space X is weakly closed if and only if it is closed.

*Proof.* Since the weak topology is weaker than the norm topology, any weakly open set is open in the norm topology, so any weakly closed set must be closed. Conversely, let $C$ be closed and convex. For $x0 \notin C$, by Theorem 10.2.6 there exist some $\Lambda \in X'$ and scalar $\alpha$ such that

$$\Lambda x0 < \alpha < \Lambda y, \forall y \in C.$$

Thus the open set $V = \{x : \Lambda x < \alpha\}$ is disjoint from $C$, so $X \setminus C$ is weakly open.

Next, consider the dual space $X0$ of a normed space $X$. We know that it is a Banach space under the operator norm. Furthermore, under the canonical identification $X$ can be viewed as a subspace of $X''$

The **weak* topology** on $X'$ is given by $\tau(X0, X)$. It is clearly Hausdorff. A local base at 0 consists of "weak* balls"

$U = \{\Lambda : |\Lambda xj| < \alpha, j = 1, \cdots, N\},$

for some $N$ and $\alpha > 0$.

The most important result in weak* topology is the following theorem.

**Theorem 10.4.4 (Alaoglu).** *The closed ball in X' is weakly* compact.*

***Proof.*** Let $P$ be the product space $\prod_{x\in X} [-||x||, ||x||]$ endowed with the product topology. By Tychonoff theorem $P$ is compact. We set up a mapping $\Phi$ from $B$, the closed unit ball in $X'$, to $P$ by setting $\Phi(\Lambda) = p$ if and only if $\Lambda x = px$, where $px$ is the projection of $P$ to $[-||x||, ||x||]$. By the definition of the product topology, its local base at $p$ is given by sets of the form

$$\{q : |qxj - pxj| < \alpha, j = 1, \cdots, N\},$$

for some $x\, j$'s and $\alpha > 0$. By comparing the weak* balls in $X'$ with this local base, we know that $\Phi$ is a homeomorphism from $(B, \tau(X0, X))$ to $P$ . To establish the theorem it suffices to show that $\Phi(B)$ is closed in $P$ , since a closed subset of a compact Hausdorff space is compact. Let $p$ be in the closure of $\Phi(B)$. We define $\Lambda x = px$. To show $p \in \Phi(B)$, we must prove that $\Lambda$ is linear and $||\Lambda|| \leq 1$.

First, we claim $\Lambda(x + y) = \Lambda x + \Lambda y$, that is, $p_{x+y} = p_x + p_y$. For, consider the open set $V$ containing $p$ given by $\{q : |qx - px|, |q_y - p_y|, |q_{x+y} - p_{x+y}| < \alpha\}$. As $p$ belongs to the closure of $\Phi(B)$, for each $\alpha > 0$, there exists some $\Lambda 1$ in $B$ in $V$ , that is, $|\Lambda_1 x - p_x|$, $|\Lambda_1 y - p_y|$, $|\Lambda 1(x + y) - p_{x+y}| < \alpha$. It follows that

$$\begin{aligned} |p_{x+y} - p_x - p_y| &\leq |p_{x+y} - \Lambda_1(x+y)| + |\Lambda_1 x + \Lambda_1 y - p_x - p_y| \\ &\leq |p_{x+y} - \Lambda_1(x+y)| + |\Lambda_1 x - p_x| + |\Lambda_1 y - p_y| \\ &< 3\alpha, \end{aligned}$$

Which implies $px+y = px + py$. Similarly, one can show that $p\alpha x = \alpha px,$ so $\Lambda$ is linear. Furthermore, from

$|\Lambda x| = |px| \leq ||x||$ we have $||\Lambda|| \leq 1$, so $\Lambda \in B$. The proof of this theorem is completed.

**CHECK YOUR PROGRESS**

3. What is weak Topology?

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4. Define **metrizable topology.**

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# 10.5 LET'S SUM UP

This is the weakest topology to make every bounded linear functional continuous. As ensured by the HahnBanach theorem, there are sufficiently many elements in $X'$ to separate points, the weak topology is Hausdorff.

In analysis Hausdorff space is preferred for many of its nice properties. For instance, a compact set is closed in a Hausdorff space. A metric space is always Hausdorff

## 10.6 KEYWORDS

**Canonical vectors** - the choice of basis of a **vector** space is arbitrary, and there is no **canonical** basis OR "without any arbitrary choices".
**Topological space** - In **topology** and related branches of **mathematics**, a **topological space** may be **defined** as a set of points, along with a set of neighbourhoods for each point, satisfying a set of axioms relating points and neighbourhoods.
**Metric space** - In **mathematics**, a **metric space** is a set together with a **metric** on the set. The **metric** is a function that defines a concept of distance between any two members of the set, which are usually called points.

## 10.7 QUESTIONS FOR REVIEW

1. Show that a weakly convergent sequence in `1 also converges strongly. This result is called Schur's theorem.

2. Show that a weakly convergent sequence in $C[a, b]$ must converge pointwisely. Given an example to show that the converse may not hold.

3. Show that in a Hilbert space $H$, $\{ x_n \} \to x$ if and only if (a) $x_n * x$ and (b) $||x_n|| \to ||x||$.

## 10.8 SUGGESTED READINGS


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## 10.9 ANSWER TO CHECK YOUR PROGRESS

1. Provide proof – 10.2.3

2. Provide explanation – 10.3

3. Provide explanation – 10.4 & statement and proof of proposition – 10.4.1

4. Provide definition – 10.4.1

# UNIT 11: NONLINEAR OPERATORS I

**STRUCTURE**



## 1.1.0 OBJECTIVE

Understand the concept of **Fixed-Point Theorems**

Comprehend different types of **Fixed-Point Theorems** and their

## 11.1 INTRODUCTION

There are three sections in this chapter. In the first section several fixed point theorems are discussed, starting from the contraction principle, Brouwer fixed-point theorem on finite dimensional space and ending on Schauder fixed point theorem. Their applications are illustrated by examples. In the next section we develop calculus on Banach space.
In this section we will discuss three widely known fixed-point theorems, starting with the fixed-point theorem established by Banach in 1920. Since its discovery, this theorem remains as one of the most frequently used results in analysis.

## 11.2 FIXED-POINT THEOREMS

For a map from a set to itself, a fixed point of this map is an element in this set which is not movedby it. Many theoretical and practical problems can be formulated as problems of finding fixed points of certain maps. The general question of solving equation, symbolically written as $f(x) = 0$, is equivalent to solving $g(x) = x$, where $g(x) \equiv f(x) + x$. Consequently any root of $f$ is a fixed point of $g$.

The setting of Banach fixed-point theorem, or contraction mapping principle, is formulated on a
complete metric space. Let $(X, d)$ be a metric space. A map $T : (X, d) \to (X, d)$ is called a *contraction* if there exists some $\gamma \in (0, 1)$ such that

$$d\left(T(x), T(y)\right) \leqslant \gamma d(x, y) \ , \ \forall x, y \in X \ .$$

It is clear that any contraction is necessarily continuous.

**Theorem 11.2.1.** Every contraction on a complete metric space has a unique fixed point.

*Proof.* Let $(X, d)$ be a complete metric space. Pick any $x_0$ from $X$ and define a sequence $\{x_n\}$ biteration: $x_n = T n(x0)$ , $n > 1$. We claim that $\{xn\}$ is a Cauchy sequence. For, we have

$$\begin{aligned} d(x_n, x_m) &= d\left(T^n(x_0)\ ,\ T^{n-m}(x_0)\right) \\ &\leqslant \gamma d\left(T^{n-1}(x_0)\ , T^{m-1}(x_0)\right) \\ &\vdots \\ &\leqslant \gamma^m d\left(T^{n-m}(x_0)\ , x_0\right) \end{aligned}$$

for any $n, m, n > m$. On the other hand, for $l \geq 1$,

$$\begin{aligned} d\left(T^l(x_0), x_0\right) &\leqslant d\left(T^l(x_0), T^{l-1}(x_0)\right) + d\left(T^{l-1}(x_0), T^{l-2}(x_0)\right) + \cdots \\ &\quad + d\left(T(x_0), x_0\right) \\ &\leqslant \left(\gamma^{l-1} + \gamma^{l-2} + \cdots + 1\right) d\left(T(x_0), x_0\right) \\ &\leqslant \frac{d\left(T(x_0), x_0\right)}{1 - \gamma} \ . \end{aligned}$$

Taking $l = n - m$, we have

$$d(x_n, x_m) \leqslant \frac{d(T(x_0), x_0)}{1-\gamma}\gamma^m \longrightarrow 0, \quad \text{as } m \to \infty,$$

hence $\{x_n\}$ is a Cauchy sequence. By completeness of the space exists.

$$z \equiv \lim_{n\to\infty} T^n(x_0)$$

By the continuity of $T$

$$T(z) = T(\lim_{n\to\infty} T^n(x_0)) = \lim_{n\to\infty} T^{n+1}(x_0) = z,$$

in other words, $z$ is a fixed point of $T$ .If $w$ is another fixed point of $T$ , then

$$0 \leqslant d(w,z) = d(T(w), T(z)) \leqslant \gamma d(w,z).$$

As $\gamma \in (0, 1)$, it forces $d(w, z) = 0$, i.e., $w = z$, so the fixed point is unique.

It is worthwhile to note that the above proof provides a constructive way to find the fixed point.
Starting from any initial point, the fixed point can be found as the limit of an iteration scheme. The contraction mapping principle has wide applications. You should have learned how it is used to establish the local solvability of the initial value problem of ordinary differential equations. Another standard application is the proof of the implicit function theorem. We shall, in the next section, shows that it can be used to prove the same theorem in the infinite dimensional setting.

Banach fixed-point theorem asserts the existence of fixed points for special maps (contractions) in a general space (a complete metric space). There are fixed-points theorems which hold for general maps in a special space. The Brouwer fixed-point theorem is the most famous one among them. It is concerned with continuous functions from the closed unit ball of the $n$-dimensional Euclidean space to itself. The complete statement was first proved by Brouwer in 1912 using homotopy, a newly invented topological concept. Over the years there are many different proofs and generalizations.

Let $B$ be a closed ball in $\mathbb{R}^n$ , $n > 1$.

**Theorem 11.2.2.** Every continuous map from B to itself has a fixed point.

This theorem is not valid when the closed ball is replaced by the open one. For instance, the map $T(x) = (1 + x)/2$ which maps (0, 1) to itself is free of fixed points.

In the following we will take the ball to be the closed unit ball centered at the origin. We begin with a computational lemma.

**Lemma 11.2.3.** Let f be twice continuously differentiable from B to B. Denote its Jacobian matrix by $Jf(x) = (\partial f^i/\partial x_j)$, i, j = 1, · · · , n. Let $c_{ij}$ be its (i, j)-th cofactor. Then for each i,

$$\sum_{j=1}^{n} \frac{\partial c_{ij}}{\partial x_j} = 0.$$

*Proof.* Without loss of generality take $i = n$. Let **g***j* be the $j$−th $(n - 1)$-column vector

$$\mathbf{g}^j = \begin{bmatrix} \dfrac{\partial f^1}{\partial x_j} \\ \vdots \\ \dfrac{\partial f^{n-1}}{\partial x_j} \end{bmatrix}.$$

We have, by the definition of the cofactor matrix,

$$c_{nj} = (-1)^{n+j} \det\left[\mathbf{g}^1, \cdots, \check{\mathbf{g}}^j, \cdots, \mathbf{g}^n\right],$$

where "v" means the $j$-th column **g***j* is removed. Note that **g**1, · · · , ˇ**g***j*, · · · , **g***n* is an $(n - 1) \times (n - 1)$- matrix. By the rule of differentiation, we have

$$\begin{aligned}\frac{\partial c_{nj}}{\partial x_j} &= (-1)^{n+j}\sum_{k<j}\det\left[\mathbf{g}^1,\cdots,\frac{\partial \mathbf{g}^k}{\partial x_j},\cdots,\breve{\mathbf{g}}^j,\cdots,\mathbf{g}^n\right] \\ &\quad+(-1)^{n+j}\sum_{k>j}\det\left[\mathbf{g}^1,\cdots,\breve{\mathbf{g}}^j,\cdots,\frac{\partial \mathbf{g}^k}{\partial x_j},\cdots,\mathbf{g}^n\right].\end{aligned}$$

Using the elementary properties of the determinant, we have

$$\begin{aligned}\frac{\partial c_{nj}}{\partial x_j} &= (-1)^{n+j}\sum_{k<j}(-1)^{k-1}\det\left[\frac{\partial \mathbf{g}^k}{\partial x_j},\cdots,\breve{\mathbf{g}}^k,\cdots,\breve{\mathbf{g}}^j,\cdots,\mathbf{g}^n\right] \\ &\quad+(-1)^{n+j}\sum_{k>j}(-1)^{k-2}\det\left[\frac{\partial \mathbf{g}^k}{\partial x_j},\cdots,\breve{\mathbf{g}}^j,\cdots,\breve{\mathbf{g}}^k,\cdots,\mathbf{g}^n\right].\end{aligned}$$

Set $\sigma_{kj}$ equal to 1 if $k < j$, to 0 if $k = j$ and to $-1$ if $k > j$. Then $\sigma_{jk} = -\sigma_{kj}$ and

$$\frac{\partial c_{nj}}{\partial x_j} = (-1)^n\sum_{k=1}^{n}(-1)^{j+k-1}\sigma_{kj}\ \det\left[\frac{\partial \mathbf{g}^k}{\partial x_j},\cdots,\breve{\mathbf{g}}^k,\cdots,\breve{\mathbf{g}}^j\cdots,\mathbf{g}^n\right].$$

$$\begin{aligned}\sum_{j=1}^{n}\frac{\partial c_{nj}}{\partial x_j} &= (-1)^n\sum_{k,j}(-1)^{j+k-1}\sigma_{kj}\det\left[\frac{\partial \mathbf{g}^k}{\partial x_j},\cdots,\breve{\mathbf{g}}^k,\cdots,\breve{\mathbf{g}}^j,\cdots,\mathbf{g}^n\right]. \\ &= (-1)^n\sum_{k,j}(-1)^{k+j-1}\sigma_{jk}\det\left[\frac{\partial \mathbf{g}^j}{\partial x_k},\cdots,\breve{\mathbf{g}}^j,\cdots,\breve{\mathbf{g}}^k,\cdots,\mathbf{g}^n\right] \\ &= (-1)^{n+1}\sum_{k,j}(-1)^{j+k-1}\sigma_{kj}\det\left[\frac{\partial \mathbf{g}^k}{\partial x_j},\cdots,\breve{\mathbf{g}}^k,\cdots,\mathbf{g}^j,\cdots,\mathbf{g}^n\right] \\ &= -\sum_{j=1}^{n}\frac{\partial c_{nj}}{\partial x_j},\end{aligned}$$

So after using $\partial \mathbf{g}^k/\partial x_j = \partial \mathbf{g}^j/\partial x_k$ in the last line, we are done.

**Proof of Theorem 11.2.2.** Let us first prove the theorem assuming that $F : B \to B$ is twice continuously differentiable. Assume on the contrary $F$ that does not have a fixed point, that's, $F(x) - x \neq 0,\ \forall x \in B$.
For each $x$, consider the equation for $\lambda$,

$$\left\|x+\lambda\big(x-F(x)\big)\right\| = 1\ ,$$

where || · || is the Euclidean norm. This is a quadratic equation; indeed, by expanding it we have

$$\|x\|^2 + 2\langle x, x - F(x)\rangle\lambda + \|x - F(x)\|^2\lambda^2 = 1 \ ,$$

where ⟨·,·⟩stands for the Euclidean inner product. There are two real roots given by

$$\lambda = \frac{\langle x, F(x) - x\rangle \pm \sqrt{\langle x, x - F(x)\rangle^2 - \|x - F(x)\|^2(\|x\|^2 - 1)}}{\|x - F(x)\|^2}$$

It is clear that the larger root $a(x)$, regarded as a function of $x$, is given by

$$a(x) = \frac{\langle x, F(x) - x\rangle + \sqrt{\langle x, x - F(x)\rangle^2 + (1 - \|x\|^2)\|x - F(x)\|^2}}{\|x - F(x)\|^2}.$$

It is readily checked that $a$ is continuously differentiable in $B$ and vanishes on $\partial B$, the boundary of $B$. (You should note that $x, F(x) - x < 0$ by the characterization of equality sign in Cauchy-Schwarz inequality).

Now, consider the one-parameter maps on $B$ to itself given by

$F(x, \lambda) = x + \lambda a(x)x - F(x)$.

We have $F(x, 0) = x$ and $F(x, 1) \in \partial B$. Consider the integral

$$I_\lambda = \int_B \det J_F(x)dx.$$

It is helpful to keep in mind that this integral gives the volume of the set F (B) in $\mathbb{R}^n$ in view of the formula of change of variables. We claim that

$$\frac{\partial I_\lambda}{\partial \lambda} = 0. \qquad (1)$$

For,

$$\begin{aligned}
\frac{\partial I_\lambda}{\partial \lambda} &= \frac{\partial}{\partial \lambda}\int_B \det J_F dx \\
&= \int_B \sum_{i,j} c_{ij}\frac{\partial^2 F^i}{\partial\lambda\partial x_j}dx \\
&= \int_B \sum \frac{\partial}{\partial x_j}\left(c_{ij}\frac{\partial F^i}{\partial \lambda}\right)dx \qquad \text{(Lemma11.3)} \\
&= \int_{\partial B} c_{ij}\frac{\partial F^i}{\partial \lambda}\nu_j dx. \qquad \text{( by the divergence theorem)}
\end{aligned}$$

Recall that the divergence theorem asserts that for any vector field v = (v1, · · · , vn) in the domain **Ω**,

$$\int_\Omega \sum_i \frac{\partial v_j}{\partial x_j} = \int_{\partial\Omega}\sum_j v_i\nu_j ds,$$

where $\nu = (\nu 1, \cdots, \nu n)$ is the unit outer normal at $\partial\Omega$, the boundary of **Ω**. Since $\partial F/\partial\lambda = a(x)(x - F(x))$ vanishes on $\partial B$, (1) follows.

From (1) we conclude that $I\lambda$ is a constant. In particular, $I_1 = I_0$. Since $F(x, 0) = x$, $I_0 = |B|$, the volume of $B$. However, on the other hand, as $F(\cdot, 1)$ maps $B$ to $\partial B$, $\det J_F(x, 1) \equiv 0$ which implies that $I_1 = 0$, contradiction holds. (To see why $\det J_F(x, 1) \equiv 0$, we may reason as follows: If $\det J_F(x_0, 1) \neq 0$ at some $x_0$. By the continuity of $\det J_F(x, 1)$ we may assume $x_0$ is located in the interior of $B$. Nonvanishing of the determinant implies that the matrix $JF(x0, 1)$ is invertible. By the inverse function theorem, the image of $F(\cdot, 1)$ would contain an open set surrounding the point $F(x0, 1)$, which would be in conflict with $F(B, 1) \subset \partial B$.)

From this contradiction we conclude that every twice continuously differentiable map from $B$ to itself has a fixed point.

Now the general case. Let $F = (F1, \cdots, Fn)$ be any continuous map from $B$ to itself. For each $Fj$, we can find a sequence of polynomials $\{P_k^j\}$ which approximate it uniformly in $B$. Therefore, the map $Fk = (P_k^1, \cdots, P_k^n,)$ is smooth from $B$ to $\mathbb{R}^n$. It is not hard to see that we can find $\lambda_k \in$

(0, 1), $\lambda_k \to 1$ such that, $G_k = \lambda_k F_k : B \to B$. Let $z_k$ by a fixed point of $G_k$. Then $G_k(z_k) = z_k$. By Bolzano-Weierstrass theorem we can extract from $z_k$ a convergent subsequence, still denoted by $\{z_k\}$ which converges to some $z$ Then

$$\begin{aligned}\|F(z) - z\| &\leqslant \|F(z) - G_k(z)\| + \|G_k(z) - G_k(z_k)\| \\ &\quad + \|G_k(z_k) - z_k\| + \|z_k - z\| \\ &\to 0.\end{aligned}$$

that's, $z$ is a fixed point for $F$ . The proof of Brouwer fixed-point theorem is completed.

Theorem 11.2.2 clearly is a topological result. It implies that every continuous map from a set homeomorphic to the closed unit ball to itself has a fixed point. In particular, this is true on compact convex sets in $\mathbb{R}^n$.

An obvious difference between the contraction mapping principle and Brouwer fixed-point theorem is the lack of uniqueness in the latter. In fact, trivial examples show that the fixed point may not be unique.

It is a usual practise in mathematics that people try to approach an important theorem from various angles and obtain different proofs. There is no exception for Brouwer fixed-point theorem. After Brouwer's topological proof, many different proofs have emerged. Our analytic proof is adapted from Dunford-Schwartz.

In functional analysis the emphasis is on infinite dimensional spaces. Can this theorem be generalized to infinite dimension? We have learned that an essential difference between finite and infinite dimensions is the loss of compactness.

It turns this phenomenon plays a role. Here is a counterexample. Consider the map $\Phi$ defined in the closed unit ball of $\ell^2$, $\{x \in \ell^2 : ||x||_2 \leq 1\}$, given by $\Phi(x) = ((1 - ||x||_2^2)^{1/2}, x1, x2, \cdots)$. It is clear that this map is continuous into the ball itself (in fact, to its boundary). However, it does not have a fixed point.

For, if $\Phi(z) = z$ for some $z$ in this ball, by equalling the components of $\Phi(z)$ and $z$ we have $z1 = z2 = z3 = \cdots$ which implies $z = (0, 0, 0 \cdots)$. But this is impossible from the first component:

$$1 - ||z||_2^2 = z_1^2$$

This example shows that continuity is not sufficient to ensure the existence of fixed points in infinite dimensional spaces. A most direct way is to restrict our attention to compact sets.

The following result is a fixed-point theorem established by Schauder in 1930.

**Theorem 11.2.4.** Let C be a non-empty compact, convex set in the Banach space X. Every continuous map from C to itself has a fixed point.

**Proof.** By compactness, for each 1/n we can cover C by finitely many balls B1/n(z1), · · · , B1/n(zN ) where the centers zj, j = 1, · · · , N, belong to C. Let $C_n$ be the convex hull of these centers, that is,

$$C_n = \{\sum_j \lambda_j z_j : \sum \lambda_j = 1,\ \lambda_j \in [0,1]\}.$$

Each $C_n$ is a compact convex set in some finite dimensional space. We define a map $Pn$ from $C$ to $C_n$ by

$$P_n(x) = \frac{\sum_j \text{dist}(x, C \setminus B_{1/n}(z_j))z_j}{\sum_j \text{dist}(x, C \setminus B_{1/n}(z_j))}.$$

It is straightforward to verify that $P_N$ is continuous and satisfies

$$||P_n(x) - x|| < \frac{1}{n},$$

in $C$. Now, consider the composite map $Pn \circ T$ and restrict it to $C_n$ to obtain a continuous map from $C_n$ to itself. Applying Brouwer fixed-point theorem to it, we obtain some $x_n$ in $C_n$ satisfying $Pn(T(xn)) = xn$.

As $C_n \subset C$ and $C$ is compact, by passing to a subsequence if necessary, we may assume $x_0 = \lim_{n\to\infty} x_n$ exists in $C$. Using the above estimate, we have

$$\|x_n - T(x_n)\| = \|P_n(T(x_n)) - T(x_n)\| < \frac{1}{n}.$$

Letting $n \to \infty$, we conclude that $kx0 - T(x0)k = 0$, that is, $x0$ is a fixed point of $T$ . The proof of Schauder fixed-point theorem is complete.

Schauder fixed-point theorem is a very common tool in the study of partial differential equations.
Let's demonstrate its power through a simple case.
Consider the ordinary differential equation

$$\frac{d^2u}{dx^2} = f(x, u, \frac{du}{dx}) \quad , \quad x \in (0, 1). \qquad (2)$$

There are two kinds of problems associated to this equation. The first one is the initial value problem, namely, we look for a solution $u(x)$ satisfies (8.3) together with the initial conditions $u(0) = a$, $u'(0) = b$ where $a$ and $b$ are prescribed values. The fundamental existence theorem of ODE's asserts that this problem has a unique solution in some interval containing 0 when $f(x, z, p)$ is sufficiently regular, for instance, it is continuously differentiable in $(x, z, p)$ near $(0, a, b)$. Alternatively, one may consider boundary value problems. For example, one may seek a solution of (11.1.3) which also satisfies the boundary conditions $u(0) = \alpha$ and $u(1) = \beta$. Boundary value problems arise from separation of variables in partial differential equations.

Here for simplicity assume the continuous function $f$ is independent of $p$ and satisfies the structural condition

$$|f(x, z)| \leqslant C_1 \left(1 + |z|^\gamma\right) \ , \ (x, z) \in [0, 1] \times \mathbb{R}, \qquad (3)$$

for some constants $C1$ and $\gamma \in (0, 1)$.

**Proposition 11.2.5.** *Under condition (11.1.3), (11.1.2) has a solution satisfying* $u(0) = u(1) = 0$.

***Proof*.** We know that (11.1.2) is equivalent to the integral equation

$$u(x) = \int G(x, y) f(y, u(y)) \, dy$$

where the integration is over [0, 1] and $G$ is the Green function of the linear problem ($q \equiv 0$). It is known that $G$, $\partial G/\partial x$, $\partial G/\partial y$ are continuous on $[0, 1]\times[0, 1]$. We choose the space

$$X = \{C[0, 1] : u(0) = u(1) = 0\}$$

with the sup-norm and define

$$Tu(x) = \int G(x, y) f(y, u(y)) dy.$$

It is clear that $T : X \to X$ is continuous. Consider the closed and convex subset

$$C = \{u \in X : ||u||_\infty, ||u'||_\infty \leq R\}.$$

As a direct consequence of Ascoli-Arzela theorem $C$ is also compact. We claim that $T$ maps $C$ into $C$ when $R$ is sufficiently large. For, from (11.1.4),

$$\begin{aligned} |Tu(x)| &\leqslant \sup_x \int |G(x, y) f(y, u(y) dy| \\ &\leqslant MC_1 \int \left(1 + |u(y)|^\gamma\right) dy \\ &\leqslant MC_1 \left(1 + R^\gamma\right), \end{aligned}$$

where $M = \sup_{x,y} |G(x, y)|$. Similarly,

$$\begin{aligned} \left|\frac{d}{dx} Tu(x)\right| &= \left|\int \frac{\partial G}{\partial x}(x, y) f(y, u(y)) dy\right| \\ &\leqslant M_1 C_1 (1 + R_0^\gamma), \end{aligned}$$

where $M1 = \sup_{x,y} |\partial G/\partial x(x, y)|$. Since $\gamma < 1$, we can choose a large $R_0$ so that

$$MC(1 + R_0^{\gamma}), \quad MC_1(1 + R_0^{\gamma}) \leqslant R_0.$$

With this choice of $R_0$, $T$ maps $C$ into itself.

Now we can apply Schauder fixed-point theorem to conclude that $T$ admits a fixed point $u \in C$. In other words,

$$u(x) = Tu(x) = \int G(x, y) f(y, u(y))dy,$$

so $u$ solves (2).

What happens when the exponent $\gamma$ in (8.4) is larger or equal to one? Things become more delicate.

We just point out that then a solution may not exist. Consider the special case

$$\begin{cases} \dfrac{d^2u}{dx^2} = -4\pi^2 u + \varphi(x) \\ u(0) = u(1) = 0. \end{cases}$$

Multiplying the equation by $\sin 2\pi x$ and then integrating over [0, 1], we obtain a necessary condition for solvability, namely,

$$\int_0^1 \varphi(x) \sin 2\pi x dx = 0.$$

In particular, this problem does not admit a solution when $\phi(x) = \sin 2\pi x$.

**CHECK YOUR PROGRESS**

1. Explain : Every contraction on a complete metric space has a unique fixed point.

2. State Brouwer fixed-point theorem

Notes

3. Provide the proof of Schauder fixed-point theorem

## 11.3 LET'S SUM UP

Starting from any initial point, the fixed point can be found as the limit of an iteration scheme. The contraction mapping principle has wide applications. It is used to establish the local solvability of the initial value problem of ordinary differential equations. Another standard application is the proof of the implicit function theorem. Schauder fixed-point theorem is a very common tool in the study of partial differential equations

## 11.4 KEYWORDS

**Structural condition - structural** properties of **mathematical** objects are usually characterized in one of two ways: either as properties expressible purely in terms of the primitive relations of **mathematical** theories, or as the properties that hold of all structurally similar **mathematical** objects

**Continuously differentiable -** A function f is said to be **continuously differentiable** if the derivative $f'(x)$ exists and is itself a **continuous** function

**Without loss of generality** - is a term used in proofs to indicate that an assumption is being made that does not introduce new restrictions to the problem.

## 11.5 QUESTIONS FOR REVIEW

1. Prove : Every contraction on a complete metric space has a unique fixed point

2.Complete the assumption and prove : f be twice continuously differentiable from B to B.

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## 11.7 ANSWER TO CHECK YOUR PROGRESS

1. Provide proof – 11.2.1

2. Provide statement and proof– 11.2.2

3. Provide statement and proof – 11.2.4

# UNIT 12: NONLINEAR OPERATORS II

**STRUCTURE**



## 12.0 OBJECTIVE

Understand the concept of **Calculus in Normed Spaces**

Comprehend various theorems of **Calculus in Normed Spaces**

Understand the application of **Calculus in Normed Spaces**

## 12.1 INTRODUCTION

Of crucial importance are the implicit and inverse function theorems which are proved by the contraction mapping principle. Finally, we discuss how to minimize a nonlinear functional over a subset in a Banach space. Recall that one valuable application of differentiation is to determine the critical points of a function. Similarly, in a function space a minimum of a certain functional is a critical point of this functional. This is a huge topic which has been split into different branches of mathematics such as the calculus of variations, optimization theory, control theory, etc. The reader may appreciate the use of convexity and weak topology in this context.

## 12.2 CALCULUS IN NORMED SPACES

The concept of differentiability has a natural generalization to infinite dimensional space. Let *F* be a map from a set *E* in the normed space *X* to another normed space *Y* and *x*0 a point in *E*. The map *F* is said to be *differentiable* at *x*0 if there exists a bounded linear operator $L \in B(X, Y)$ such that

$$\|f(x) - f(x_0) - L(x - x_0)\| = o(\|x - x_0\|), \quad \text{as } x \to x_0,$$

in other words

$$\lim_{x \to x_0} \frac{\|f(x) - f(x_0) - L(x - x_0)\|}{\|x - x_0\|} = 0.$$

The linear operator is called the *Fr`echet derivative* or simply the derivative of *F* at *x*0. The map *F* is called differentiable on *E* if it is differentiable at every point of *E*. In this case, the derivative is a bounded linear operator depending on $x \in E$ and usually is denoted by *F'*(*x*) or *DF* (*x*). We call *F* a *C*1-*map* if $x \mapsto F'(x)$ is continuous from *E* to *B*(*X*, *Y*). Let's consider two examples.

First, let $X = C^1[0, 1]$ and $Y = C[0, 1]$ under the *C*1- and sup-norms respectively. The map

$$F(u)(t) = t \sin u(t) + (u'(t))^2$$

maps *X* to *Y*. To find its derivative we need to determine the linear operator *F 0*(*u*) such that

$$\lim_{w \to u} \frac{\|F(w) - F(u) - F'(u)(w - u)\|_\infty}{\|w - u\|_{C^1}} = 0.$$

Setting $w = u + \varepsilon\phi$ in the above, we see that in case *F*'(*u*) exists, we must have

$$F'(u)\varphi(t) = \lim_{\varepsilon \to 0} \frac{F(u(t) + \varepsilon\varphi(t)) - F(u(t))}{\varepsilon},$$

at each $u(t)$ and $\phi(t)$. Applying the chain rule in the variable $\varepsilon$, we readily obtain

$$F'(u)\varphi = t\cos(u(t))\ \varphi(t) + 2u'(t)\varphi'(t).$$

It is now a direct check using the definition that this expression indeed gives the derivative of $F$ . Observe that it is linear on $\phi$ but nonlinear in $u$.

Second, consider the nonlinear functional $S : C1(\Omega) \to$ R given by

$$S(u) = \int_\Omega \sqrt{1 + |\nabla u(x)|^2} dx,$$

where Ω is a bounded domain in R$n$. This functional gives the surface area of the hypersurface $\{(x, u(x))\}$ over Ω. Proceeding as above, its derivative is given by

$$DS(u)\varphi = \int_\Omega \frac{\langle \nabla u, \nabla \varphi \rangle}{\sqrt{1 + |\nabla u|^2}},$$

a bounded linear functional on $C1(\Omega)$ under the $C1$-norm.

Here are some elementary properties of differentiability.

**Proposition 12.2.1 .** Let X, Y and Z be normed spaces and E ⊂ X and N ⊂ Y.

(i) Let F, G : E → Y be differentiable at x. Then αF + βG is differentiable at x and

$$D(\alpha F + \beta G)(x) = \alpha DF(x) + \beta DG(x).$$

(ii) Let F : E → N and G : N → Z be differentiable at x and F (x) respectively. Then G ◦ F is differentiable at x and

$$D(G \circ F)(x) = DG(F(x))DF(x).$$

When $F$ is differentiable (resp. $C1$) in $E$, $f(E) \subset N$ and $G$ is differentiable (resp. $C1$) in $N$, this proposition shows that $G \circ F$ is differentiable (resp. $C1$) in $E$.

Let $\phi$ be a continuous map from the interval $[a, b]$ to the Banach space $X$. When $X$ = R, we can define the **Riemann integral** of $\phi$ over $[a, b]$. Likewise the same thing can be done in a Banach space. The old definition works, namely, $\phi$ is *integrable* on $[a, b]$ if there exists an element $z$ in $X$ such that, for every $\varepsilon > 0$, there exists some $\delta > 0$, so that

$$\Big\| \sum_j \varphi(t_j)\Delta t_j - z \Big\|_X < \varepsilon,$$

for any partition $P$ of $[a, b]$ whose length is less than $\delta$. The number $z$ is called the *integral* of $\phi$ and will be denoted by $\int_a^b \varphi\ (t)dt$.. Be careful it is an element in $X$. Same as in the one dimensional case, any continuous map on $[a, b]$ is integrable.

**Proposition 12.2.2.** Let ϕ : [a, b] → X be continuous where X is a Banach space.

(i) There holds

$$\Big\| \int_a^b \varphi(t)dt \Big\| \le \int_a^b \|\varphi(t)\| dt,$$

(ii) For every Λ ∈ X0,

$$\int_a^b (\Lambda\varphi)(t)dt = \Lambda\Big( \int_a^b \varphi(t)dt \Big).$$

(iii) If ϕ is a C1-map, then

$$\varphi(b) - \varphi(a) = \int_a^b \varphi'(t)dt.$$

*Proof.* (i) and (ii) follow directly from definition. For (iii), we observe that if $\phi(b) - \phi(a)$ is not equal to $\int_a^b \varphi'(t)dt$, Hahn-Banach theorem tells us that there exists some $\Lambda 1 \in X'$such that

$$\Lambda_1 (\varphi(b) - \varphi(a)) \neq \Lambda_1 \left( \int_0^1 \varphi'(t)dt \right).$$

However, by linearity and (ii), contradiction holds.

$$\begin{aligned} \Lambda_1 (\varphi(b) - \varphi(a)) &= (\Lambda_1 \varphi)(b) - (\Lambda_1 \varphi)(a) \\ &= \int_a^b (\Lambda_1 \varphi)'(t)dt \\ &= \int_a^b \Lambda_1 \varphi'(t)dt \\ &= \Lambda_1 \int_a^b \varphi'(t)dt, \end{aligned}$$

Now we come to the fundamental inverse function theorem. Roughly speaking, it tells us that a map is locally invertible at a particular point if its linearization at the same point is invertible.

**Theorem 12.2.3.** Let F : U → Y be a C1-map where X and Y are Banach spaces and U is open in X. Suppose that F $(x_0) = y_0$ and F'$(x_0)$ is invertible. There exist open sets V and W containing x0 and y0 respectively such that the restriction of F on V is a bijection onto W with a C1-inverse.

Recall that a bounded linear operator is invertible if its inverse exists and is bounded.

*Proof.* Without loss of generality take $x_0, y_0 = 0$. First we would like to show that there is a unique solution for the equation $F(x) = y$ for $y$ near 0. We shall use the contraction mapping principle to achieve our goal. For a fixed $y$, define the map in $U$ by

$$T(x) = L^{-1} (Lx - F(x) + y)$$

**Notes**

where $L = F'(0)$. It is clear that any fixed point of $T$ is a solution to $F(x) = y$. We have

$$\begin{aligned} \|T(x)\| &\leqslant \|L^{-1}\| \, \|F(x) - Lx - y\| \\ &\leqslant \|L^{-1}\| \left( \|F(x) - Lx\| + \|y\| \right) \\ &\leqslant \|L^{-1}\| \left( \circ(\|x\|) + \|y\| \right) . \end{aligned}$$

We can find a small $\rho_0$ such that

$$\|L^{-1}\| \circ (\|x\|) \leq \frac{1}{4}\|x\|, \quad \forall x, \quad \|x\| \leq \rho_0. \quad (1)$$

Then for for each $y$ in $BR(0)$, $\|L^{-1}\|R \leq \rho_0/2$, $T$ maps $\overline{\beta\rho_0}$ to itself. Moreover, for $x1$, $x2$ in $B\,\rho_0\,(0)$, we have

$$\begin{aligned} \|T(x_2) - T(x_1)\| &= \left\|L^{-1}\left(F(x_2) - Lx_2 - y\right) - L^{-1}\left(F(x_1) - Lx_1 - y\right)\right\| \\ &\leqslant \|L^{-1}\| \, \|F(x_2) - F(x_1) - F'(0)(x_2 - x_1)\| \\ &\leqslant \|L^{-1}\| \left\| \int_0^1 F'\left(x_1 + t(x_2 - x_1)\right)(x_2 - x_1)dt - F'(0)(x_2 - x_1) \right\|, \end{aligned}$$

where we have applied Proposition 12.1.1 (ii) to $\phi(t) = F(x1 + t(x2 - x1))$. Since $F\ 0$ is continuous in $U$, by further restricting $\rho_0$ we may assume

$$\|F'(x) - F'(0)\| < \frac{1}{2\left(\|L^{-1}\| + 1\right)}, \quad \forall x \in B_{\rho_0}(0) .$$

Consequently,

$$\begin{aligned} \|T(x_2) - T(x_1)\| &\leqslant \|L^{-1}\| \frac{1}{2(1 + \|L^{-1}\|)} \|x_2 - x_1\| \\ &< \frac{1}{2}\|x_2 - x_1\| . \end{aligned}$$

We have shown that $T : \overline{\beta\rho_0(0)} \to \overline{\beta\rho_0(0)}$ is a contraction. By the contraction mapping principle, there is a unique fixed point for $T$, in other words, for each $y$ in the ball $BR(0)$ there is a unique point $x$ in $B\rho_0$

(0) solving $F(x) = y$. Defining $G : BR(0) \to B\rho_0(0) \subset X$ by setting $G(y) = x$, $G$ is inverse to $F$.

Next, we claim that $G$ is continuous. In fact, for $G(y_i) = x_i$, $i = 1, 2$,

$$\begin{aligned}
\|G(y_2) - G(y_1)\| &= \|x_2 - x_1\| \\
&= \|T(x_2) - T(x_1)\| \\
&\leqslant \|L^{-1}\| \, \| (\circ\|x_2 - x_1\| + \|y_2 - y_1\|) \\
&\leq \|L^{-1}\| \, \| (\circ\|x_2\| + \circ\|x_1\| + \|y_2 - y_1\|) \\
&\leq \frac{1}{2}\|x_2 - x_1\| + \|L^{-1}\|\|y_2 - y_1\| \\
&= \frac{1}{2}\|G(y_2) - G(y_1)\| + \|L^{-1}\|\|y_2 - y_1\|,
\end{aligned}$$

which, by (1), implies that's, $G$ is continuous on $BR(0)$.

$$||G(y_2) - G(y_1)|| \leq 2||L^{-1}||||y_2 - y_1|| \,, \qquad (2)$$

Finally, let's show that $G$ is a $C1$-map in $BR(0)$. In fact, for $y1$, $y1 + y$ in $BR(0)$, using

$$\begin{aligned}
y &= F(G(y_1 + y)) - F(G(y_1)) \\
&= \int_0^1 F'(G(y_1) + t(G(y_1 + y) - G(y_1))dt \; (G(y_1 + y) - G(y_1)),
\end{aligned}$$

$$G(y_1 + y) - G(y_1) = F'^{-1}(G(y_1))y + R,$$

where $R$ is given by

$$F'^{-1}(G(y_1)) \int_0^1 \Big(F'(G(y_1)) - F'(G(y_1) + t(G(y_1 + y) - G(y_1))\Big)(G(y_1 + y) - G(y_1))dt.$$

As $G$ is continuous and $F$ is $C1$, we have

$$G(y_1 + y) - G(y_1) - F'^{-1}(G(y_1))y = \circ(1)(G(y_1 + y) - G(y_1))$$

for small $y$. Using (2), we see that

$$G(y_0 + y) - G(y_0) - F'^{-1}(G(y_0))y = o(\|y\|) \ ,$$

as $||y|| \to 0$. We conclude that $G$ is differentiable with derivative equal to $F'^{-1}(G(y0))$. The proof of the inverse function theorem is completed by taking $W = BR(0)$ and $V = F^{-1}(W)$.

**Remark.** Under the setting of Theorem 12.1.3, what happens if the invertibility of $F'(x0)$ is replaced by surjectivity? Well, assume that $X$ is the direct sum of $X1$ and $X2 \equiv \ker F'(x0)$. Then the following conclusion holds: There exist $V1$, $V2$ and $W$ open subsets of $X1$, $X2$ and $Y$ respectively and $C1$-map $G : W \to V1 \times V2$ such that for each $x2$ in $X2$, $G(\cdot, x2)$ is the inverse to $F(\cdot, x2)$.

Next we deduce the implicit function theorem from the inverse function theorem. In fact, these two theorems are equivalent; in the exercise you are asked to give a self-contained proof of the implicit function theorem and deduce the inverse function theorem from the implicit function theorem.

**Theorem 12.2.4.** Consider C1-map F : U → Z where U is an open set in the Banach spaces X × Y . Suppose $(x_0, y_0) \in U$ and F (x0, y0) = 0. If $Fy(x_0, y_0)$ is invertible from Y to Z, then there exist an open subset U1 × V1 of U containing (x0, y0) and a C1-map $\phi : U1 \to V1$, $\phi(x_0) = y_0$, such that $F(x, \phi(x)) = 0$, $\forall x \in U1$.
Moreover, if $\psi$ is a C1-map from U2, an open set containing x0, to Y satisfying $F(x, \psi(x)) = 0$ and $\psi(x_0) = y_0$, then $\psi$ equals to $\phi$ in $U1 \cap U2$.

The notation $F\,y(x_0, y_0)$ stands for the "partial derivative" of F in y, that is, the derivative of $F$ at $y0$ while $x0$ is fixed as a constant.

***Proof.*** Consider $\Phi : U \to X \times N$ given by

$$\Phi(x, y) = (x, F(x, y)).$$

By assumption

$$\Phi'(x_0, y_0)(x, y) = (x, Fy(x_0, y_0)(x, y))$$

is invertible from $X \times Y$ to $X \times Z$. By the inverse function theorem, there exists some

$$\Psi = (\Psi 1, \Psi 2) : U1 \times N1 \rightarrow U$$

which is inverse to Φ. For every $(x, z) \in U1 \times N1$, we have

$$\Phi(\Psi 1(x, z), \Psi 2(x, z)) = (x, z),$$

which immediately implies

$$\Psi 1(x, z) = x, \text{ and } F((\Psi 1(x, z), \Psi 2(x, z)) = z.$$

In particular, taking $z = 0$ gives

$$(x, 0) = \Phi(\Psi(x, 0)) = (x, F(x, \Psi 2(x, 0)), \ \forall x \in U1,$$

so the function $\phi(x) \equiv \Psi 2(x, 0)$ satisfies our requirement. The uniqueness assertion can be easily established and is left to the reader.

The implicit function theorem is indispensable in analysis. You will encounter many of its applications as you go along. Here we give a very simple one about the multiplicity of solutions to differential equations.

Consider the boundary value problem

$$\begin{cases} \frac{d^2u}{dx^2} = -\lambda u + g(u), \\ u(0) = u(1) = 0, \end{cases}$$

where $\lambda$ is a given number and $g(y)$ is a function satisfying $g(0) \equiv 0$. Clearly the zero function is a trivial solution of this problem. An interesting question is, could it admit another solution? Taking the special case $g \equiv 0$ where the equation can be solved explicitly, we see that it has a nonzero solution if and only if $\lambda = n^2\pi^2$, $n \in \mathbb{N}$. Indeed, the solutions are given by $u(x) = c \sin n\pi x,$ where $c$ is an arbitrary nonzero constant.

A value at which nontrivial solutions exist arbitrarily near the trivial solution is called a bifurcation point. In this problem, every number $n^2\pi^2$ is a bifurcation point. In the general case, the zero function is still a trivial solution. We would like to know which $\lambda$ is a bifurcation point. To this end, we take $X$, $Y$, and $Z$ respectively to be R, $\{u \in C2[0, 1] : u(0) = u(1) = 0\}$, and $C[0, 1]$ and

$$F(\lambda, u) = u'' + \lambda u + g(u).$$

We have $F(\lambda, 0) = 0$ and

$$F_y(\lambda, 0)v = v'' + \lambda v + g_y(0)v.$$

Clearly $Fy(\lambda, 0)$ is invertible if and only if $\lambda$ is not equal to $n^2\pi^2 - gy(0)$, $n \in$ N. By the implicit function theorem, there exists an open set containing the zero function which does not contain any additional solution to the problem. Hence values not equal to $n^2\pi^2 - gy(0)$ cannot be bifurcation points. What happens when $\lambda$ is equal to $n^2\pi^2 - g_y(0)$? This is bifurcation theory. More information is required from $g$ to obtain a conclusion.

The technique in the proof of the inverse function theorem can be used to establish a nonlinear version of the open mapping theorem.

**Theorem 12.2.5.** Let F be a $C^1$-map from U to Y where U is open in X and X, Y are Banach spaces. Suppose that F 0(x) maps X onto Y for every x in U. Then F (U) is open in Y.

**Lemma 12.2.6.** *Let $T \in B(X, Y)$ be surjective where X and Y are Banach spaces. There exists a constant C such that*

$$\inf\{kx - zk : z \in \ker T\} \leq CkTxk, \ \forall x \in X$$

Proof. Consider the quotient Banach space X ˜ = X/ ker T under the norm

$$\|\tilde{x}\| = \inf\{\|x - z\| : \ z \in \ker T\}.$$

The induced map $\widetilde{T} : \widetilde{X} \to Y$ given by $\widetilde{T}\,\widetilde{x} = T\,x,\ x \in \widetilde{x}$ is a bounded linear operator onto $Y$. By Banach inverse mapping theorem $T\tilde{}$ is invertible, that is, and the lemma follows.

$$\|\tilde{x}\| \leq C\|\tilde{T}\tilde{x}\|.$$

**Proof of Theorem 12.2.5.** It suffices to show that if $F'(x_0)(X) = Y$ where $x_0 \in U$, there exist balls $B\,\rho(x_0)$ and $BR(y_0)$, $y_0 = F\,(x_0)$, such that $\subset BR(y_0)F\,(B\rho(x_0))$.

With $\rho$ and $R$ both small to be specified, for any fixed $y$ in $BR(y0)$ we define a sequence $\{xn\}$ in $B\,\rho/2(x_0)$ as follows. First, find $x'_{n+1} \in X$ such that

$$Tx'_{n+1} = Tx_n - (F(x_n) - y), \quad T = F'(x_0).$$

Of course such point exists as $T$ is onto. As

$$\inf \|x_n - (x'_{n+1} + z)\| \leq C\|Tx_n - Tx_{n+1}\|,$$

for all $z \in \ker T$ by the above lemma, we could modify $x0\ n{+}1$ by some element in $\ker T$ to get $x_{n+1}$ satisfying

$$\|x_{n+1} - x_n\| \leq (C+1)\|Tx_n - Tx_{n+1}\|.$$

Starting from $n = 0$, we have

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq (C+1)\|F(x_n) - y\| \\ &= (C+1)\|F(x_n) - F(x_{n-1}) - Tx_n + Tx_{n-1}\| \\ &\leq (C+1) \circ (\|x_n - x_{n-1}\|). \end{aligned}$$

We can choose a small $\rho$ such that

$$\|x_{n+1} - x_n\| \leq \frac{1}{2}\|x_n - x_{n-1}\|,$$

as long as $\{x_n\}$ stays in $B_{\rho/2}(x_0)$. From the above estimate we have

$$\|x_{n+1} - x_0\| \leq \sum_{j=0}^{n} \frac{1}{2^j} \|x_1 - x_0\| \leq 2\|y_0 - y\|.$$

By choosing $R < \rho/4$, for every $y \in B_R(y_0)$, $\{x_n\} \subset B_{\rho/2}(x_0)$. Moreover,

$$\|x_n - x_m\| \leq \frac{1}{2^m} \|x_{n-m} - x_0\| \leq \frac{\rho}{2^{m+1}} \to 0,$$

as $n > m$, $m \to \infty$. By completeness there is some $x$ in $B_\rho(x_0)$ such that $x = \lim_{n\to\infty} x_n$. From $Tx = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Tx_{n+1}$ we deduce that $x$ solves $F(x) = y$.

Let $F : U \to Y$ be differentiable where $U$ is open in $X$. Its derivative $F'(x)$ belongs to $B(X, Y)$. It is *twice differentiable* at $x$ if $F' : U \to B(X, Y)$ is differentiable at $x$. The second (Fréchet) derivative at $x$, denoted by $F''(x)$ or $D^2F(x)$, belongs to $B(X, B(X, Y))$. $F$ is a $C^2$-map if $x \mapsto F''(x)$ is continuous from $U$ to $B(X, B(x, Y))$.

There is a natural way to identify the space $B(X, B(X, Y))$ with the multi-linear space $M_2(X, Y)$ where $X$ and $Y$ are normed spaces. A map $T : X \times X \to Y$ is a *bilinear form* from $X$ to $Y$ if $T(x_1, x_2)$ is linear in $x_1$ (resp. $x_2$) while $x_2$ (resp. $x_1$) is fixed. All continuous, bilinear maps from $X \times X$ to $Y$ form a vector space $M_2(X, Y)$. For any such map $T$, define

$$\|T\| = \sup_{x,y} \{\|T(x_1, x_2)\|_Y : \|x_1\|_X, \|x_2\|_X \leqslant 1\} .$$

It is readily checked that $(M_2(X, Y), \|\cdot\|)$ forms a normed space, and it is complete when $Y$ is complete.

Given $T \in B(X, B(X, Y))$, define

$$\hat{T}(x_1, x_2) = T(x_1)x_2 , \qquad x_1, x_2 \in X .$$

It is routine to verify that $T \mapsto \hat{T}$ b established a norm-preserving isomorphism from $B(X, B(X, Y))$ to $M2(X, Y)$. Under this isomorphism we may identify $B(X, B(X, Y))$ with $M2(X, Y)$. It follows that the second derivative $F''(x)$ may be regarded as a bilinear form with value in $Y$. In fact, the following proposition shows that it is symmetric.

**Proposition 12.2.7.** Let F be a C2-map from U to Y where U is open in X and X, Y are normed spaces. Then

$$F''(x)(x_1, x_2) = F''(x)(x_2, x_1)\ , \qquad \forall x \in U\ ,\ x_1, x_2 \in X\ .$$

***Proof.*** For $x1, x2 \in U$ and $\varepsilon 1, \varepsilon 2$ small, $x + \varepsilon 1 x1 + \varepsilon 2 x2 \in U$. Consider the $C2$-function $\phi$ given by

$$\phi(\varepsilon 1, \varepsilon 2) = \Lambda\,(F\,(x + \varepsilon 1 x1 + \varepsilon 2 x2))$$

where $\Lambda$ is in $Y'$. By the chain rule

$$\frac{\partial \varphi}{\partial \varepsilon_1} = \Lambda F'(x + \varepsilon_1 x_1 + \varepsilon_2 x_2) x_1\ ,$$

$$\frac{\partial \varphi}{\partial \varepsilon_2} = \Lambda F'(x + \varepsilon_1 x_1 + \varepsilon_2 x_2) x_2\ ,$$

So, at $(\varepsilon 1, \varepsilon 2) = (0, 0)$,

$$\begin{aligned} \frac{\partial^2 \varphi}{\partial \varepsilon_2 \partial \varepsilon_1} &= \Lambda F''(x + \varepsilon_1 x_1 + \varepsilon_2 x_2)(x_1, x_2) \\ &= \Lambda F''(x)(x_1, x_2) \\ \frac{\partial^2 \varphi}{\partial \varepsilon_1 \partial \varepsilon_2} &= \Lambda F''(x)(x_2, x_1)\ . \end{aligned}$$

The desired result follows from the relation $\partial^2 \phi / \partial \varepsilon_2 \partial \varepsilon_1 = \partial^2 \phi / \partial \varepsilon_1 \partial \varepsilon_2$.

Similarly one can define the $m$-th derivative of $F$ and identify it with an $m$-linear function. Same as in this proposition, $F\,(m)(x)(x1, \cdots, xm)$ is symmetric in $(x1, \cdots, xm)$ when $F$ is a $Cm$-map.

Notes

**CHECK YOUR PROGRESS**

1. Explain calculus in Normed Spaces

2. Define bifurcation point

3. What do you understand by: $\inf\{\|x - z\| : z \in \ker T\} \leq C\|Tx\|, \ \forall x \in X$

## 12.3 LET'S SUM UP

The technique in the proof of the inverse function theorem can be used to establish a nonlinear version of the open mapping theorem. It may help to formulate and prove a version of Taylor's expansion theorem in the infinite dimensional setting.

## 12.4 KEYWORDS

**Bijection-** In **mathematics**, a **bijection**, **bijective** function, one-to-one correspondence, or invertible function, is a function between the elements of two sets, where each element of one set is paired with exactly one element of the other set, and each element of the other set is paired with exactly one element of the first set.

**Unique point -** a **point** refers usually to an element of some set called a space, unique is meant to capture the notion of a **unique** location in Euclidean space.

**Analysis -** is a branch of **mathematics** which studies continuous changes and includes the theories of integration, differentiation, measure, limits, analytic functions and infinite series. It is the systematic study of real and complex-valued continuous functions

## 12.5 QUESTIONS FOR REVIEW

1. **Prove the proposition:** Let X, Y and Z be normed spaces and $E \subset X$ and $N \subset Y$.

(i) Let $F, G : E \to Y$ be differentiable at x. Then $\alpha F + \beta G$ is differentiable at x and

$$D(\alpha F + \beta G)(x) = \alpha DF(x) + \beta DG(x).$$

(ii) Let $F : E \to N$ and $G : N \to Z$ be differentiable at x and F (x) respectively. Then $G \circ F$ is differentiable at x and

$$D(G \circ F)(x) = DG(F(x))DF(x).$$

2.State the background of theorem 12.2.4 with proof.

## 12.6 SUGGESTED READINGS


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## 12.7 ANSWER TO CHECK YOUR PROGRESS

1. Provide explanation – 12.2

2. Provide definition –12.2.4

3. Provide statement and proof–12.2.6

# UNIT 13: FOURIER ANALYSIS

**STRUCTURE**



## 13.0 OBJECTIVE

Understand the concept of Convolutions on Commutative Groups

Comprehend the Characters of Commutative Groups

Understand the application of Fourier Transform on Commutative Groups & The Schwartz space of smooth rapidly decreasing functions

Enumerate the Fourier Integral

## 13.1 INTRODUCTION

In mathematics, **Fourier analysis** is the study of the way general functions may be represented or approximated by sums of simpler trigonometric functions.

Fourier analysis grew from the study of Fourier series, and is named after Joseph Fourier, who showed that representing a function as a sum of trigonometric functions greatly simplifies the study of heat transfer.

Today, the subject of Fourier analysis encompasses a vast spectrum of mathematics. In the sciences and engineering, the process of decomposing a function into oscillatory components is often called Fourier analysis, while the operation of rebuilding the function from these pieces is known as **Fourier synthesis**

Our purpose is to map the commutative algebra of convolutions to a commutative algebra of functions with point-wise multiplication. To this end we first represent elements of the group as operators of multiplication.

## 13.2 CONVOLUTIONS ON COMMUTATIVE GROUPS

Let $G$ be a commutative group, we will use + sign to denote group operation, respectively the inverse elements of $g \in G$ will be denoted $-g$. We assume that $G$ has a Hausdorff topology such that operations $(g_1,g_2) \mapsto g_1+g_2$ and $g \mapsto -g$ are continuous maps. We also assume that the topology is *locally compact*, that is the group neutral element has a neighbourhood with a compact closure.

**Example :** Our main examples will be as follows:

1. G=$\mathbb{Z}$ the group of integers with operation of addition and the discrete topology (each point is an open set).
2. G=$\mathbb{R}$ the group of real numbers with addition and the topology defined by open intervals.
3. G=T the group of Euclidean rotations the unit circle in $\mathbb{R}^2$ with the natural topology. Another realisations of the same group:
   - Unimodular complex numbers under multiplication.
   - Factor group $\mathbb{R}/\mathbb{Z}$, that is addition of real numbers modulo 1.

There is a homomorphism between two realisations given by $z=e^{2\pi i t}$, $t \in [0,1)$, $|z|=1$.

We assume that $G$ has a regular Borel measure which is invariant in the following sense.

**Definition 13.2.1** *Let* μ *be a measure on a commutative group G,* μ *is called* invariant *(or* Haar measure*) if for any measurable X and any* $g \in G$ *the sets* $g+X$ *and* $-X$ *are also measurable and* $\mu(X)=\mu(g+X)=\mu(-X)$.

Such an invariant measure exists if and only if the group is locally compact, in this case the measure is uniquely defined up to the constant factor.

**Exercise :** Check that in the above three cases invariant measures are:

- $G=\mathbb{Z}$, the invariant measure of X is equal to number of elements in X.
- $G=\mathbb{R}$ the invariant measure is the Lebesgue measure.
- $G=T$ the invariant measure coincides with the Lebesgue measure.

**Definition 13.2.2** A *convolution* of two functions on a commutative group $G$ with an invariant measure $\mu$ is defined by:

$$(f_1 * f_2)(x)=\int_G f_1(x-y)\, f_2(y)\, d\mu(y)=\int_G f_1(y)\, f_2(x-y)\, d\mu(y).$$

**Theorem 13.2.3** If $f_1, f_2 \in L_1(G,\mu)$, then the integrals in exist (above definition 13.1.2) for almost every $x \in G$, the function $f_1 * f_2$ is in $L_1(G,\mu)$ and $\|f_1 * f_2\| \leq \|f_1\| \cdot \|f_2\|$.

**Proof.** If $f_1, f_2 \in L_1(G,\mu)$ then by Fubini's Thm. 50 the function $\varphi(x,y)=f_1(x)*f_2(y)$ is in $L_1(G\times G, \mu\times \mu)$ and $\|\varphi\|=\|f_1\|\cdot \|f_2\|$.

Let us define a map $\tau: G\times G \rightarrow G\times G$ such that $\tau(x,y)=(x+y,y)$. It is measurable (send Borel sets to Borel sets) and preserves the measure $\mu\times\mu$. Indeed, for an elementary set $C=A\times B \subset G\times G$ we have:

$$(\mu\times\mu)(\tau(C)) = \int_{G\times G} \chi_{\tau(C)}(x,y)\, d\mu(x)\, d\mu(y)$$

$$= \int_{G\times G} \chi_C(x-y,y)\, d\mu(x)\, d\mu(y)$$

$$= \int_G \left[ \int_G \chi_C(x-y,y)\, d\mu(x) \right] d\mu(y)$$

$$= \int_B \mu(A+y)\, d\mu(y) = \mu(A)\times \mu(B) = (\mu\times\mu)(C).$$

we have an isometric isomorphism of $L_1(G\times G,\mu\times \mu)$ into itself by the formula:

$$T\varphi(x,y)=\varphi(\tau(x,y))=\varphi(x-y,y).$$

If we apply this isomorphism to the above function $\varphi(x,y)=f_1(x)*f_2(y)$ we shall obtain the statement.

**Definition 13.2.*4*** Denote by *S(k)* the map *S(k): f*$\mapsto$ *k*f* which we will call *convolution operator* with the *kernel k*.

**Corollary 13.2.5** *If* $k\in L_1(G)$ *then the convolution* S(k) *is a bounded linear operator on* $L_1(G)$.

**Theorem 13.2.6** Convolution is a commutative, associative and distributive operation. In particular $S(f_1)S(f_2)=S(f_2)S(f_1)=S(f_1*f_2)$.
**Proof.** Direct calculation using change of variables.

It follows from Thm13.2.3 that convolution is a closed operation on $L_1(G)$ and has nice properties due to Thm. 13.1.6. We fix this in the following definition.

**Definition 13.2.7** $L_1(G)$ *equipped with the operation of convolution is called* convolution algebra $L_1(G)$.
The following operators of special interest.

**Definition 13.2.8** An operator of *shift* $T(a)$ acts on functions by $T(a): f(x) \mapsto f(x+a)$.

**Lemma 13.2.9** An operator of shift is an isometry of $L_p(G)$, $1 \leq p \leq \infty$.

**Theorem 13.2.10** Operators of shifts and convolutions commute:

$$T(a)(f_1 * f_2) = T(a)f_1 * f_2 = f_1 * T(a)f_2,$$

*or*

$$T(a)S(f) = S(f)T(a) = S(T(a)f).$$

**Proof.** Just another calculation with a change of variables.

**Remark 13.2.11** Note that operator of shifts $T(a)$ provide a *representation* of the group $G$ by linear isometric operators in $L_p(G)$, $1 \leq p \leq \infty$. A map $f \mapsto S(f)$ is a *representation* of the convolution algebra *There is a useful relation between support of functions and their convolutions.*

**Lemma 13.2.12** *For any* $f_1, f_2 \in L_1(G)$ *we have:*

$$supp(f_1 * f_2) \subset supp(f_1) + supp(f_2).$$

**Proof.** If $x \notin supp(f_1) + supp(f_2)$ then for any $y \in supp(f_2)$ we have $x - y \notin supp(f_1)$. Thus for such $x$ convolution is the integral of the identical zero.

# 13.3 CHARACTERS OF COMMUTATIVE GROUPS

**Definition 13.3.1** *A* character $\chi$: G→ T *is a continuous homomorphism of an abelian topological group* G *to the group* T *of unimodular complex numbers under multiplications:*

$$\chi(x+y)=\chi(x)\chi(y).$$

Note, that a character is an eigenfunction for a shift operator $T(a)$ with the eigenvalue $\chi(a)$. Furthermore, if a function $f$ on $G$ is an eigenfunction for all shift operators $T(a)$, $a\in G$ then the collection of respective eigenvalues $\lambda(a)$ is a homomorphism of $G$ to $\mathbb{C}$ and $f(a)=\alpha\,\lambda(a)$ for some $\alpha\in\mathbb{C}$. Moreover, if $T(a)$ act by isometries on the space containing $f(a)$ then $\lambda(a)$ is a homomorphism to T.

**Lemma 13.3.2** The product of two characters of a group is again a character of the group. If $\chi$ is a character of $G$ then $\chi^{-1}=\chi$ is a character as well.

**Proof.** Let $\chi_1$ and $\chi_2$ be characters of $G$. Then:

$$\begin{aligned}\chi_1(gh)\chi_2(gh) &= \chi_1(g)\chi_1(h)\chi_2(g)\chi_2(h)\\ &= (\chi_1(g)\chi_2(g))(\chi_1(h)\chi_2(h))\in \mathrm{T}.\end{aligned}$$

**Definition 13.3.3** The *dual group* $\hat{G}$ is collection of all characters of $G$ with operation of multiplication. *The dual group becomes a topological group with the* uniform convergence on compacts*: for any compact subset* $K\subset G$ *and any* $\varepsilon>0$ *there is* $N\in\mathbb{N}$ *such that* $|\chi_n(x)-\chi(x)|<\varepsilon$ *for* all $x\in K$ and $n>N$.

**Example** *If* $G=\mathbb{Z}$ *then any character* $\chi$ *is defined by its values* $\chi(1)$ *since*

$$\chi(n)=[\chi(1)]^n. \quad (2)$$

Since $\chi(1)$ can be any number on $T$ we see that $\hat{\mathbb{Z}}$ is parametrised by $T$.

**Theorem 13.3.4** The group $\hat{\mathbb{Z}}$ is isomorphic to $T$.

**Proof.** The correspondence from the above example is a group homomorphism. Indeed if $\chi_z$ is the character with $\chi_z(1)=z$, then $\chi_{z1}\chi_{z2}=\chi_{z1\,z2}$. Since ℤ is discrete, every compact consists of a finite number of points, thus uniform convergence on compacts means point-wise convergence. The equation (2) shows that $\chi_{zn}\rightarrow\chi_z$ if and only if $\chi_{zn}(1)\rightarrow\chi_z(1)$, that is $z_n\rightarrow z$.

**Theorem 13.3.5** *The group* $T^\wedge_$ *is isomorphic to* ℤ*.*

**Proof.** For every $n\in\mathbb{Z}$ define a character of T by the identity

$$\chi_n(z)=z^n, \quad z\in T. \quad (3)$$

We will show that these are the only characters . The isomorphism property is easy to establish. The topological isomorphism follows from discreteness of $T^\wedge_$. Indeed due to compactness of T for $n\neq m$:

$$\max_{z\in T} \left|\chi_n(z)-\chi_m(z)\right|^2=\max_{z\in T}\left|1-\Re\, z^{m-n}\right|^2=2^2=4.$$

Thus, any convergent sequence $(n_k)$ have to be constant for sufficiently large $k$, that corresponds to a discrete topology on ℤ.

The two last Theorems are an illustration to the following general statement.

**Principle 13.3.6 (Pontryagin's duality)** *For any locally compact commutative topological group G the natural map G→ Ĝ, such that it maps g∈ G to a character* $f_g$ *on Ĝ by the formula:*

$f_g(\chi)=\chi(g), \quad \chi\in\hat{G}, \quad (4)$

*is an isomorphism of topological groups.*

**Remark:**

1. The principle is not true for commutative group which are not locally compact.
2. Note the similarity with an embedding of a vector space into the second dual.

**Notes**

In particular, the Pontryagin's duality tells that the collection of all characters contains enough information to rebuild the initial group.

**Theorem 13.3.7** *The group* $\hat{\mathbb{R}}$ *is isomorphic to* $\mathbb{R}$.

**Proof.** For $\lambda\in\mathbb{R}$ define a character $\chi_\lambda\in\hat{\mathbb{R}}$ by the identity

$$\chi_\lambda(x)=e^{2\pi i\lambda x},\qquad x\in\mathbb{R}.\qquad (5)$$

Moreover any smooth character of the group $G=(\mathbb{R}, +)$ has the form (5). Indeed, let $\chi$ be a smooth character of $\mathbb{R}$.
Put $c=\chi'(t)|_{t=0}\in\mathbb{C}$.
Then

$$\chi'(t)=c\chi(t) \text{ and } \chi(t)=e^{ct}.$$

We also get $c\in i\mathbb{R}$ and any such $c$ defines a character. Then the multiplication of characters is:

$$\chi_1(t)\chi_2(t)=e^{c_1t}e^{c_2t}=e^{(c_2+c_1)t}.$$

So we have a group isomorphism.

For a generic character we can apply first the *smoothing technique* and reduce to the above case.
Let us show topological homeomorphism. If $\lambda_n\to\lambda$ then $\chi_{\lambda n}\to\chi_\lambda$ uniformly on any compact in $\mathbb{R}$ from the explicit formula of the character. Reverse, let $\chi_{\lambda n}\to\chi_\lambda$ uniformly on any interval. Then $\chi_{\lambda n-\lambda}(x)\to 1$ uniformly on any compact, in particular, on [0,1]. But

Thus $\lambda_n\to\lambda$.

**Corollary 13.3.8** Any character of the group T has the form (3).

**Proof.** Let $\chi\in\hat{T}$, consider $\chi_1(t)=\chi(e^{2\pi i t})$ which is a character of $\mathbb{R}$. Thus $\chi_1(t)=e^{2\pi i\lambda t}$ for some $\lambda\in\mathbb{R}$. Since $\chi_1(1)=1$ then $\lambda=n\in\mathbb{Z}$. Thus $\chi_1(t)=e^{2\pi i n t}$, that is $\chi(z)=z^n$ for $z=e^{2\pi i t}$.

**Remark 27** Although $ℝ^_$ is isomorphic to ℝ there is no a canonical form for this isomorphism (unlike for ℝ→ $ℝ^_$). Our choice is convenient for the Poisson formula below, however some other popular definitions are $\lambda \to e^{i\lambda x}$ or $\lambda \to e^{-i\lambda x}$.

We can unify the previous three Theorem into the following statement.

**Theorem 13.3.9** *Let* $G=\mathbb{R}^n\times \mathbb{Z}^k\times \mathrm{T}^l$ *be the direct product of groups. Then the dual group is* $\hat{G}=\mathbb{R}^n\times \mathrm{T}^k\times \mathbb{Z}^l$.

**CHECK YOUR PROGRESS**

1. Define Convolution

---

2. State Pontryagin's duality

---

## 13.4 FOURIER TRANSFORM ON COMMUTATIVE GROUPS

**Definition 13.4.1** Let $G$ be a locally compact commutative group with an invariant measure $\mu$. For any $f\in L_1(G)$ define the *Fourier transform* $f$ by

$$f(\chi)=\int_G f(x)\,\bar{\chi}(x)\,d\mu(x),\qquad \chi\in\hat{G}. \quad (6)$$

That is the Fourier transform f is a function on the dual group Ĝ.

The important properties of the Fourier transform are captured in the following statement.

**Theorem 13.4.2** Let G be a locally compact commutative group with an invariant measure μ. The Fourier transform maps functions from $L_1(G)$ to continuous bounded functions on $\hat{G}$. Moreover, a convolution is transformed to point-wise multiplication:

$$(f_1 * f_2)^\wedge(\chi) = f_1(\chi) \cdot f_2(\chi), \quad (7)$$

a shift operator $T(a)$, $a \in G$ is transformed in multiplication by the character $f_a \in \hat{G}$:

$$(T(a)f)^\wedge(\chi) = f_a(\chi) \cdot f(\chi), \quad f_a(\chi) = \chi(a) \quad (8)$$

*and multiplication by a character* $\chi \in \hat{G}$ *is transformed to the shift* $T(\chi^{-1})$*:*

$$(\chi \cdot f)^\wedge(\chi_1) = T(\chi^{-1}) f(\chi_1) = f(\chi^{-1}\chi_1). \quad (9)$$

**Proof.** Let $f \in L_1(G)$. For any $\varepsilon > 0$ there is a compact $K \subset G$ such that $\int_{G \setminus K} |f| \, d\mu < \varepsilon$. If $\chi_n \to \chi$ in $\hat{G}$, then we have the uniform convergence of $\chi_n \to \chi$ on $K$, so there is $n(\varepsilon)$ such that for $k > n(\varepsilon)$ we have $|\chi_k(x) - \chi(x)| < \varepsilon$ for all $x \in K$. Then

$$\left| f(\chi_n) - f(\chi) \right| \le \int_K |f(x)| \left| \chi_n(x) - \chi(x) \right| d\mu(x) + \int_{G \setminus K} |f(x)| \left| \chi_n(x) - \chi(x) \right| d\mu(x)$$

$$\le \varepsilon \| f \| + 2\varepsilon.$$

Thus f is continuous. Its boundedness follows from the integral estimations. Algebraic maps (7)–(9) can be obtained by changes of variables under integration.

$$(f_1 * f_2)^\wedge(\chi) = \int_G \int_G f_1(s) f_2(t-s)\, ds\, \bar{\chi}(t)\, dt$$

$$= \int_G \int_G f_1(s)\, \overline{\chi(s)}\, f_2(t-s)\, \bar{\chi}(t-s)\, ds\, dt$$

$$= f_1(\chi) f_2(\chi).$$

## 13.5 THE SCHWARTZ SPACE OF SMOOTH RAPIDLY DECREASING FUNCTIONS

We say that a function $f$ is *rapidly decreasing* if $\lim_{x\to\pm\infty} |x^k f(x)| = 0$ for any $k \in \mathbb{N}$.

**Definition 13.5.1** *The* Schwartz space *denoted by S or space of rapidly decreasing functions on Rn is the space of infinitely differentiable functions such that:*

$$S = \left\{ f \in C^\infty(\mathbb{R}) : \sup_{x \in \mathbb{R}} \left| x^\alpha f^{(\beta)}(x) \right| < \infty \quad \forall\, \alpha, \beta \in \mathbb{N} \right\}. \quad (10)$$

**Example:** An example of a rapidly decreasing function is the Gaussian $e^{-\pi x^2}$.

It is worth to notice that $S \subset L_p(\mathbb{R})$ for any $1 < p < \infty$. Moreover, $S$ is dense in $L_p(\mathbb{R})$, for $p=1$ this can be shown in the following steps (other values of $p$ can be done similarly but require some more care). First we will show that $S$ is an ideal of the convolution algebra $L_1(\mathbb{R})$.

**Notes**

**CHECK YOUR PROGRESS**

**3.** Explain Fourier Transform on Commutative Groups

4. What is Schwartz space?

# 13.6 FOURIER INTEGRAL

**Definition 13.6.1** *We define the* Fourier integral *of a function* $f \in L_1(\mathbb{R})$ *by*

$$f(\lambda)=\int_{\mathbb{R}} f(x)\, e^{-2\pi i \lambda x}\, dx. \qquad (11)$$

**Lemma 13.6.2** *The Fourier integral* f *of* $f \in L_1(\mathbb{R})$ *has zero limits at* $-\infty$ *and* $+\infty$.

**Proof.** Take $f$ the indicator function of $[a,b]$. Then $f(\lambda)=1/-2\pi i \lambda\,(e^{-2\pi i a} - e^{-2\pi i b})$, $\lambda \neq 0$. Thus $\lim_{\lambda \to \pm\infty} f(\lambda)=0$. By continuity from the previous Lemma this can be extended to the closure of step functions, which is the space $L_1(\mathbb{R})$ by Lem. 17.

**Lemma 13.6.3** *If f is absolutely continuous on every interval and* $f' \in L_1(\mathbb{R})$*, then*

$$(f')^{\wedge}=2\pi i \lambda\, f.$$

More generally:

$$(f^{(k)})^{\wedge}=(2\pi i \lambda)^k f. \qquad (12)$$

**Proof.** A direct demonstration is based on integration by parts, which is possible because assumption in the Lemma.

It may be also interesting to mention that the operation of differentiation $D$ can be expressed through the shift operatot $T_a$:

$$D = \lim_{\Delta t \to 0} \frac{T_{\Delta t} - I}{\Delta t}. \qquad (13)$$

By the formula (8), the Fourier integral transforms $1/\Delta t(T_{\Delta t} - I)$ into $1/\Delta t(\chi_\lambda(\Delta t) - 1)$. Providing we can justify that the Fourier integral commutes with the limit, the last operation is multiplication by $\chi'_\lambda(0) = 2\pi i \lambda$.

**Corollary 13.6.4** *If* $f^{(k)} \in L_1(\mathbb{R})$ *then*

$$\left| \hat{f} \right| = \frac{\left| (f^{(k)})^{\wedge} \right|}{\left| 2\pi\lambda \right|^k} \to 0 \quad \text{as } \lambda \to \infty,$$

that is $\hat{f}$ decrease at infinity faster than $|\lambda|^{-k}$.

**Lemma 13.6.5** Let f(x) and xf(x) are both in $L_1(\mathbb{R})$, then $\hat{f}$ is differentiable and

$$\hat{f}' = (-2\pi i x f)^{\wedge}.$$

More generally

$$\hat{f}^{(k)} = ((-2\pi i x)^k f)^{\wedge}. \qquad (14)$$

**Proof.** There are several strategies to prove these results, all having their own merits:

1. The most straightforward uses the differentiation under the integration sign.

Notes

2. We can use the intertwining property (9) of the Fourier integral and the connection of derivative with shifts (13).
3. Using the inverse Fourier integral (see below), we regard this Lemma as the dual to the Lemma 13.5.3.

**Corollary 13.6.6** The Fourier transform of a smooth rapidly decreasing function is a smooth rapidly decreasing function.

**Corollary 13.6.7** The Fourier integral of the Gaussian $e^{-\pi x2}$ is $e^{-\pi \lambda 2}$.

**Proof.** Note that the Gaussian $g(x)=e^{-\pi x2}$ is a unique (up to a factor) solution of the equation $g'+2\pi x g=0$. Then, by
Lemmas 13.5.3and 13.5.5, its Fourier transform shall satisfy to the equation $2\pi i \lambda \hat{g}+ i\hat{g}'=0$. Thus, $\hat{g}=c\cdot e^{-\pi \lambda 2}$ with a constant factor $c$, its value 1 can be found from the classical integral $\int_{\mathbb{R}} e^{-\pi x2} dx=1$ which represents $\hat{g}(0)$.
The relation (12) and (13) allows to reduce many partial differential equations to algebraic one, to finish the solution we need the inverse of the Fourier transform.
**Definition 13.6.8** *We define the* inverse Fourier transform *on* $L_1(\mathbb{R})$*:*

$$f(\lambda)=\int_{\mathbb{R}} f(x)\, e^{2\pi i \lambda x}\, dx. \qquad (15)$$

We can notice the formal correspondence f(λ)=f(−λ)=f(λ), which is a manifestation of the group duality $ℝ^_$=ℝ for the real line. This immediately generates analogous results from Lem. 38 to Cor. 44 for the inverse Fourier transform.

**Theorem 13.6.9** The Fourier integral and the inverse Fourier transform are inverse maps. That is, if $g=f$ then $f=\check{g}$.

**Proof.** The outline of the proof is as follows. Using the intertwining relations (12) and (14), we conclude the composition of Fourier integral and the inverse Fourier transform commutes both with operator of multiplication by $x$ and differentiation. Then we need a result, that any operator commuting with multiplication by $x$ is an operator of multiplication by a function $f$. For this function, the commutation with differentiation implies $f'=0$, that is $f=const$. The value of this constant can be evaluated by a Fourier transform on a single function, say the Gaussian $e^{-\pi x 2}$ .

The above Theorem states that the Fourier integral is an invertible map. For the Hilbert space $L_2(\mathbb{R})$ we can show a stronger property—its unitarity.

**Theorem 47 (Plancherel identity)** *The Fourier transform extends uniquely to a unitary map* $L_2(\mathbb{R})\to L_2(\mathbb{R})$*:*

$$\int_{\mathbb{R}} \left|f\right|^2 dx = \int_{\mathbb{R}} \left|\hat{f}\right|^2 d\lambda. \qquad (16)$$

**Proof.** The proof will be done in three steps: first we establish the identity for smooth rapidly decreasing functions, then for $L_2$ functions with compact support and finally for any $L_2$ function.

1. Take $f_1$ and $f_2 \in S$ be smooth rapidly decreasing functions and $g_1$ and $g_2$ be their Fourier transform

$$\int_{\mathbb{R}} f_1(t)\,\bar{f}_2(t)\,dt = \int_{\mathbb{R}} \int_{\mathbb{R}} g_1(\lambda)\, e^{2\pi i \lambda t}\, d\lambda\; \bar{f}_2(t)\, dt$$

$$= \int_{\mathbb{R}} g_1(\lambda) \int_{\mathbb{R}} e^{2\pi i \lambda t}\; \bar{f}_2(t)\, dt\, d\lambda$$

$$= \int_{\mathbb{R}} g_1(\lambda)\; \bar{g}_2(\lambda)\, d\lambda$$

Put $f_1=f_2=f$ (and therefore $g_1=g_2=\hat{f}$) we get the identity $\int|f|^2\,dx=\int|\hat{f}|^2\,d\lambda$. The same identity (16) can be obtained from the property $(f_1 f_2)\hat{}=\hat{f}_1 * \hat{f}_2$, cf. (7), or explicitly:

$$\int_{\mathbb{R}} f_1(x)\, f_2(x)\, e^{-2\pi i\lambda x}\, dx = \int_{\mathbb{R}} f_1(t)\, f_2(\lambda - t)\, dt.$$

1. Now, substitute $\lambda=0$ and $f_2=f_1$ (with its corollary $f_2(t)=f_1(-t)$) and obtain (16).
2. Next let $f\in L_2(\mathbb{R})$ with a support in $(-a,a)$ then $f\in L_1(\mathbb{R})$ as well, thus the Fourier transform is well-defined. Let $f_n\in S$ be a sequence with support on $(-a,a)$ which converges to $f$ in $L_2$ and thus in $L_1$. The Fourier transform $g_n$ converges to $g$ uniformly and is a Cauchy sequence in $L_2$ due to the above identity. Thus $g_n\to g$ in $L_2$ and we can extend the Plancherel identity by continuity to $L_2$ functions with compact support.
3. The final bit is done for a general $f\in L_2$ the sequence

$$f_n(x)=\begin{cases} f(x), & \text{if } |x|<n, \\ 0, & \text{otherwise;} \end{cases}$$

of truncations to the interval $(-n,n)$. For $f_n$ the Plancherel identity is established above, and $f_n\to f$ in $L_2(\mathbb{R})$. We also build their Fourier images $g_n$ and see that this is a Cauchy sequence in $L_2(\mathbb{R})$, so $g_n\to g$.

If $f\in L_1\cap L_2$ then the above $g$ coincides with the ordinary Fourier transform on $L_1$.

We note that Plancherel identity and the Parseval's identity are cousins—they both states that the Fourier transform $L_2(G)\to L_2(\hat{G})$ is an isometry for $G=\mathbb{R}$ and $G=\mathrm{T}$ respectively. They may be combined to state the unitarity of the Fourier transform on $L_2(G)$ for the group $G=\mathbb{R}^n\times \mathbb{Z}^k\times \mathrm{T}^l$ cf.

## 13.7 LET'S SUM UP

The technique in the proof of the inverse function theorem can be used to establish a nonlinear version of the open mapping theorem. It may help to formulate and prove a version of Taylor's expansion theorem in the infinite dimensional setting.

Plancherel identity and the Parseval's identity combined to state the unitarity of the Fourier transform

## 13.8 KEYWORDS

Truncating a Number. A method of approximating a decimal number by dropping all decimal places past a certain point without rounding

**Extend: Extension** (predicate logic), the set of tuples of values that satisfy the predicate. **Extension** (semantics), the set of things to which a property applies

**Convergence, in mathematics**, property (exhibited by certain infinite series and functions) of approaching a limit more and more closely as an argument (variable) of the function increases or decreases or as the number of terms of the series increases.

## 13.9 QUESTIONS FOR REVIEW

1. Suppose that the function $f_1$ is compactly supported and k times continuously differentiate in $\mathbb{R}$, and that the function $f_2$ belongs to $L_1(\mathbb{R})$. Prove that the convolution $f_1 * f_2$ has continuous derivatives up to order k.

2. Check that

If X is a compact set then the topology of uniform convergence on compacts and the topology uniform convergence on X coincide.

3. For any g∈ S and $f \in L_1(\mathbb{R})$ with compact support their convolution f*g belongs to S. Define the family of functions $g_t(x)$ for t>0 in S by scaling the Gaussian:

$$g_t(x) = \frac{1}{t} e^{-\pi (x/t)^2}.$$

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## 13.11 ANSWER TO CHECK YOUR PROGRESS

1. Provide definition– 13.2.2

2. Provide statement –13.3.6

3. Provide definition–13.4.1

4. Provide definition and example – 13.5.1

# UNIT 14: MEASURE THEORY

**STRUCTURE**



## 14.0 OBJECTIVE

Understand the Basic Measure Theory

Enumerate Extension of Measures

Comprehend the Characters of Commutative Groups

Understand the Complex-Valued Measures and Charges

How to Construct Measures, Products

## 14.1 INTRODUCTION

A measurable space is a set S, together with a non-empty collection, S, of subsets of S, satisfying the following two conditions: 1. For any A, B in the collection S, the set1 A – B is also in S. 2. For any A1 , A2 , … S, Ai S. The elements of S are called measurable sets. These two conditions are summarised by saying that the measurable sets are closed under taking finite differences and countable unions.

# 14.2 BASIC MEASURE THEORY

**Definition 14.2.1** *Let X be a set. A* σ-algebra *R on X is a collection of subsets of X, written* $R \subseteq 2^X$, *such that*

1. $X \in R$;
2. *if* $A, B \in R$, *then* $A \setminus B \in R$;
3. *if* $(A_n)$ *is any sequence in R, then* $\cup_n A_n \in R$.

Note, that in the third condition we admit any *countable* unions. The usage of "σ" in the names of σ -algebra and σ-ring is a reference to this. If we replace the condition by if $(A_n)_1^m$ is any finite family in $R$, then $\cup_{n=1}^m A_n \in R$;

then we obtain definitions of an *algebra*.

For a σ-algebra $R$ and $A, B \in R$, we have

$$A \cap B = X \setminus \left[ X \setminus (A \cap B) \right] = X \setminus \left[ (X \setminus A) \cup (X \setminus B) \right] \in R.$$

Similarly, $R$ is closed under taking (countably) infinite intersections.

If we drop the first condition from the definition of (σ-)algebra (but keep the above conclusion from it!) we got a (σ-)*ring*, that is a (σ-)ring is closed under (countable) unions, (countable) intersections and subtractions of sets.

Sets $A_k$ are *pairwise disjoint* if $A_n \cap A_m = \emptyset$ for $n \neq m$. We denote the union of pairwise disjoint sets by $\sqcup$, e.g. $A \sqcup B \sqcup C$.

It is easy to work with a vector space through its basis. For a ring of sets the following notion works as a helpful "basis".

**Definition 14.2.2** *A* semiring *S of sets is a collection such that*

1. *it is closed under intersection;*
2. *for* $A, B \in S$ *we have* $A \setminus B = C_1 \sqcup \dots \sqcup C_N$ *with* $C_k \in S$.

Again, any non-empty semiring contain the empty set.

**Example** *The following are semirings but not rings:*

1. The collection of intervals [a,b) on the real line;
2. The collection of all rectangles $\{ a \leq x < b, c \leq y < d \}$ on the plane.

As the intersection of a family of σ-algebras is again a σ-algebra, and the power set $2^X$ is a σ-algebra, it follows that given any collection $D \subseteq 2^X$, there is a σ-algebra R such that $D \subseteq R$, such that if S is any other σ-algebra, with $D \subseteq S$, then $R \subseteq S$. We call R the σ-algebra generated by D.

We introduce the symbols $+\infty$, $-\infty$, and treat these as being "extended real numbers", so $-\infty < t < \infty$ for $t \in \mathbb{R}$. We define $t+\infty = \infty$, $t\infty = \infty$ if $t>0$ and so forth. We do not (and cannot, in a consistent manner) define $\infty - \infty$ or $0\infty$.

**Definition 14.2.3** *A* measure *is a map* $\mu: R \to [0,\infty]$ *defined on a (semi-)ring (or* σ*-algebra)* R*, such that if* $A = \sqcup_n A_n$ *for* $A \in R$ *and a finite subset* $(A_n)$ *of* R*, then* $\mu(A) = \sum_n \mu(A_n)$. *This property is called* additivity *of a measure.*

In analysis we are interested in infinities and limits, thus the following extension of additivity is very important.

**Definition 14.2.4** In terms of the previous definition we say that $\mu$ is *countably additive* (or *σ-additive*) if for any countable infinite family $(A_n)$ of pairwise disjoint sets from $R$ such that $A = \sqcup_n A_n \in R$ we have $\mu(A) = \sum_n \mu(A_n)$. If the sum diverges, then as it will be the sum of positive numbers, we can, without problem, define it to be $+\infty$.

**Example**

1. Fix a point $a \in \mathbb{R}$ and define a measure $\mu$ by the condition $\mu(A)=1$ if $a \in A$ and $\mu(A)=0$ otherwise.
2. For the ring obtained in Exercise 5 from semiring $S$ in Example 1 define $\mu([a,b))=b-a$ on $S$. This is a measure, and we will show its $\sigma$-additivity.

3. For ring obtained in Exercise 5 from the semiring in Example 2, define $\mu(V)=(b-a)(d-c)$ for the rectangle $V=\{ a\leq x < b, c\leq y < d \} S$. It will be again a $\sigma$-additive measure.
4. Let $X=\mathbb{N}$ and $R=2^{\mathbb{N}}$, we define $\mu(A)=0$ if $A$ is a finite subset of $X=\mathbb{N}$ and $\mu(A)=+\infty$ otherwise. Let $A_n=\{n\}$, then $\mu(A_n)=0$ and $\mu(\sqcup_n A_n)=\mu(\mathbb{N})=+\infty\neq \sum_n \mu(A_n)=0$. Thus, this measure is *not* $\sigma$-additive.

**Definition 14.2.5** A measure $\mu$ is *finite* if $\mu(A)<\infty$ for all $A\in R$. A measure $\mu$ is *$\sigma$-finite* if $X$ is a union of countable number of sets $X_k$, such that for any $A\in R$ and any $k\in \mathbb{N}$ the intersection $A\cap X_k$ is in $R$ and $\mu(A\cap X_k)<\infty$.

**Proposition 14.2.6** Let μ be a σ-additive measure on a σ-algebra R. Then:

1. If $A,B\in R$ with $A\subseteq B$, then $\mu(A)\leq\mu(B)$ [we call this property "monotonicity of a measure"];
2. If $A,B\in R$ with $A\subseteq B$ and $\mu(B)<\infty$, then $\mu(B\setminus A) = \mu(B) - \mu(A)$;
3. If $(A_n)$ is a sequence in R, with $A_1 \subseteq A_2 \subseteq A_3 \subseteq\cdots$. Then

$$\lim_{n\to\infty} \mu(A_n) = \mu\left( \cup_n A_n \right).$$

If $(A_n)$ is a sequence in $R$, with $A_1 \supseteq A_2 \supseteq A_3 \supseteq\cdots$. If $\mu(A_m)<\infty$ for some $m$, then

$$\lim_{n\to\infty} \mu(A_n) = \mu\left( \cap_n A_n \right)$$

**Proof.** The two first properties are easy to see. For the third statement, define

$$A=\cup_n A_n,\ B_1=A_1 \text{ and } B_n=A_n\setminus A_{n-1},\ n>1.$$

Then

$$A_n=\sqcup_{k=1}^{n} B_n \text{ and } A=\sqcup_{k=1}^{\infty}B_n.$$

Using the σ-additivity of measures

$$\mu(A)=\sum_{k=1}^{\infty}\mu(B_k) \text{ and } \mu(A_n)=\sum_{k=1}^{n}\mu(B_k).$$

From the theorem in real analysis that any monotonic sequence of real numbers converges (recall that we admit $+\infty$ as limits' value) we have

$$\mu(A)=\sum_{k=1}^{\infty}\mu(B_k)=\lim_{n\to\infty}\sum_{k=1}^{n}\mu(B_k) = \lim_{n\to\infty}\mu(A_n).$$

The last statement can be shown similarly.

## 14.3 EXTENSION OF MEASURES

From now on we consider only finite measures; an extension to σ-finite measures will be done later.

**Proposition 14.3.1** *Any measure $\mu'$ on a semiring $S$ is uniquely extended to a measure $\mu$ on the generated ring $R(S)$. If the initial measure was σ-additive, then the extension is σ-additive as well.*

**Proof.** If an extension exists it shall satisfy $\mu(A)=\sum_{k=1}^{n}\mu'(A_k)$, where $A_k\in S$. We need to show for this definition two elements:

1. Consistency, i.e. independence of the value from a presentation of $A\in R(S)$ as $A=\sqcup_{k=1}^{n} A_k$, where $A_k\in S$. For two different presentation $A=\sqcup_{j=1}^{n} A_j$ and $A=\sqcup_{k=1}^{m} B_k$ define $C_{jk}=A_j\cap B_k$, which will be pair-wise disjoint. By the additivity of $\mu'$ we have $\mu'(A_j)=\sum_k\mu'(C_{jk})$ and $\mu'(B_k)=\sum_j\mu'(C_{jk})$. Then

$$\sum_j \mu'(A_j)=\sum_j\sum_k \mu'(C_{jk})=\sum_k\sum_j \mu'(C_{jk})=\sum_k \mu'(B_k).$$

Additivity. For $A=\sqcup_{k=1}^{n} A_k$, where $A_k\in R(S)$ we can present $A_k=\sqcup_{j=1}^{n(k)} C_{jk}$, $C_{jk}\in S$.

Thus $A=\sqcup_{k=1}^{n}\sqcup_{j=1}^{n(k)} C_{jk}$ and:

$$\mu(A)=\sum_{k=1}^{n}\sum_{j=1}^{n(k)}\mu'(C_{jk})=\sum_{k=1}^{n}\mu(A_k).$$

Finally, show the σ-additivity. For a set $A=\sqcup_{k=1}^{\infty} A_k$, where $A$ and $A_k\in R(S)$, find presentations $A=\sqcup_{j=1}^{n} B_j$, $B_j\in S$ and $A_k=\sqcup_{l=1}^{m(k)} B_{lk}$, $B_{lk}\in S$. Define $C_{jlk}=B_j \cap B_{lk}\in S$, then $B_j=\sqcup_{k=1}^{\infty}\sqcup_{l=1}^{m(k)} C_{jlk}$ and $A_k= \sqcup_{j=1}^{n} \sqcup_{l=1}^{m(k)} C_{jlk}$ Then, from σ-additivity of μ′:

$$\mu(A) = \sum_{j=1}^{n} \mu'(B_j)= \sum_{j=1}^{n} \sum_{k=1}^{\infty} \sum_{l=1}^{m(k)} \mu'(C_{jlk})= \sum_{k=1}^{\infty} \sum_{j=1}^{n} \sum_{l=1}^{m(k)} \mu'(C_{jlk}) = \sum_{k=1}^{\infty} \mu(A_k),$$

where we changed the summation order in series with non-negative terms.

**Definition 14.3.2** *Let S be a semi-ring of subsets in X, and* μ *be a measure defined on S. An* outer measure $\mu^*$ *on X is a map* $\mu^*:2^X\to[0,\infty]$ *defined by:*

$$\mu^*(A)=\inf\left\{ \sum_k \mu(A_k), \text{ such that } A\subseteq \cup_k A_k, \quad A_k\in S \right\}.$$

**Proposition 14.3.3** An outer measure has the following properties:

1. $\mu^*(\emptyset)=0$;
2. if $A\subseteq B$ then $\mu^*(A)\le\mu^*(B)$, this is called monotonicity of the outer measure;
3. if $(A_n)$ is any sequence in $2^X$, then $\mu^*(\cup_n A_n) \le \sum_n \mu^*(A_n)$.

The final condition says that an outer measure is *countably sub-additive*. Note, that an outer measure may be not a measure in the sense of Defn. 6 due to a luck of additivity.

**Example:** *The* Lebesgue outer measure *on* ℝ *is defined out of the measure, that is, for* $A\subseteq\mathbb{R}$, *as*

$$\mu^*(A) = \inf\left\{ \sum_{j=1}^{\infty} (b_j-a_j) : A\subseteq \cup_{j=1}^{\infty}[a_j,b_j) \right\}.$$

We make this definition, as intuitively, the “length”, or measure, of the interval $[a,b)$ is $(b-a)$.

For example, for outer Lebesgue measure we have $\mu^*(A)=0$ for any countable set, which follows, as clearly $\mu^*(\{x\})=0$ for any $x\in\mathbb{R}$.

**Lemma 14.3.4** *Let* $a<b$. *Then* $\mu^*([a,b])=b-a$.

**Proof.** For $\epsilon>0$, as $[a,b] \subseteq [a,b+\epsilon)$, we have that $\mu^*([a,b])\leq (b-a)+\epsilon$. As $\epsilon>0$, was arbitrary, $\mu^*([a,b]) \leq b-a$.

To show the opposite inequality we observe that $[a,b)\subset[a,b]$ and $\mu^*[a,b)$ $=b-a$ (because $[a,b)$ is in the semi-ring) so $\mu^*[a,b]\geq b-a$ by 2

Our next aim is to construct measures from outer measures. We use the notation $A\Delta B=(A\cup B)\setminus(A\cap B)$ for *symmetric difference of sets*.

**Definition 14.3.5** *Given* an outer measure $\mu^*$ defined by a measure $\mu$ on a semiring $S$, we define $A\subseteq X$ to be *Lebesgue measurable* if for any $\varepsilon>0$ there is a finite union $B$ of elements in $S$ (in other words: $B\in R(S)$ ), such that $\mu^*(A\,\Delta\,B)<\varepsilon$ .

Obviously all elements of $S$ are measurable. An alternative definition of a measurable set is due to Carathéodory.

**Definition 14.3.6** Given an outer measure $\mu^*$, we define $E\subseteq X$ to be *Carathéodory measurable* if

$$\mu^*(A) = \mu^*(A\cap E) + \mu^*(A\setminus E),$$

*for any* $A\subseteq X$.

As $\mu^*$ is sub-additive, this is equivalent to

$$\mu^*(A) \geq \mu^*(A\cap E) + \mu^*(A\setminus E) \qquad (A\subseteq X),$$

as the other inequality is automatic.

Suppose now that the ring $R(S)$ is an algebra (i.e., contains the maximal element $X$). Then, the outer measure of any set is finite, and the following theorem holds:

**Theorem 14.3.7 (Lebesgue)** Let $\mu^*$ be an outer measure on $X$ defined by a semiring $S$, and let $L$ be the collection of all Lebesgue measurable sets for $\mu^*$. Then $L$ is a $\sigma$-algebra, and if $\mu'$ is the restriction of $\mu^*$ to $L$, then $\mu'$ is a measure. Furthermore, $\mu'$ is $\sigma$-additive on $L$ if $\mu$ is $\sigma$-additive on $S$.

**Proof.** Clearly, $R(S)\subset L$. Now we show that $\mu^*(A)=\mu(A)$ for a set $A\in R(S)$. If $A\subset\cup_k A_k$ for $A_k \in S$, then $\mu(A)\leq \sum_k \mu(A_k)$, taking the infimum we get $\mu(A)\leq\mu^*(A)$. For the opposite inequality, any $A\in R(S)$ has a disjoint representation $A=\sqcup_k A_k$, $A_k\in S$, thus $\mu^*(A)\leq \sum_k \mu(A_k)=\mu(A)$. Now we will show that $R(S)$ is an incomplete metric space, with the measure μ being uniformly continuous functions. Measurable sets make the completion of $R(S)$ with μ being continuation of $\mu^*$ to the completion by continuity.

Define a distance between elements $A$, $B\in L$ as the outer measure of the symmetric difference of $A$ and $B$: $d(A,B)=\mu^*(A\Delta B)$. Introduce equivalence relation $A\sim B$ if $d(A,B)=0$ and use the inclusion for the triangle inequality:

$$A \Delta B \subseteq (A\Delta C) \cup (C\Delta B)$$

Then, by the definition, Lebesgue measurable sets make the closure of $R(S)$ with respect to this distance.

We can check that measurable sets form an algebra. To this end we need to make estimations, say, of $\mu^*((A_1\cap A_2) \Delta (B_1\cap B_2))$ in terms of $\mu^*(A_i\Delta B_i)$. A demonstration for any finite number of sets is performed through mathematical inductions. The above two-sets case provide both: the base and the step of the induction.

Now, we show that $L$ is σ-algebra. Let $A_k\in L$ and $A=\cup_k A_k$. Then for any $\varepsilon>0$ there exists $B_k\in R(S)$, such that $\mu^*(A_k\triangle B_k)<\varepsilon/2^k$. Define $B=\cup_k B_k$. Then

$$\left(\cup_k A_k\right) \Delta \left(\cup_k B_k\right) \subset \cup_k \left(A_k \Delta B_k\right)$$

implies $\mu^*(A \Delta B)<\varepsilon$.

We cannot stop at this point since $B=\cup_k B_k$ may be not in $R(S)$. Thus, define $B'_1=B_1$ and $B'_k=B_k \setminus \cup_{i=1}^{k-1} B_i$, so $B'_k$ are pair-wise disjoint. Then $B=\sqcup_k B'_k$ and $B'_k \in R(S)$. From the convergence of the series there is $N$ such that $\sum_{k=N}^{\infty}\mu(B'_k)<\varepsilon$ . Let $B'=\cup_{k=1}^{N} B'_k$, which is in $R(S)$. Then $\mu^*(B\Delta B')\leq \varepsilon$ and, thus, $\mu^*(A\Delta B')\leq 2\varepsilon$.

To check that $\mu^*$ is measure on $L$ we use the following

**Lemma 14.3.8** $|\mu^*(A)-\mu^*(B)|\leq \mu^*(A \Delta B)$, *that is* $\mu^*$ *is uniformly continuous in the metric* $d(A,B)$.

**Proof.**[Proof of the Lemma] Use inclusions $A\subset B\cup(A \Delta B)$ and $B\subset A\cup(A \Delta B)$.

To show additivity take $A_{1,2}\in L$ , $A=A_1\sqcup A_2$, $B_{1,2}\in R(S)$ and $\mu^*(A_i\Delta B_i)<\varepsilon$. Then $\mu^*(A \Delta (B_1\cup B_2))<2\varepsilon$ and $|\mu^*(A) - \mu^*(B_1\cup B_2)|<2\varepsilon$. Thus $\mu^*(B_1\cup B_2)=\mu(B_1\cup B_2)=\mu(B_1)+\mu(B_2)-\mu(B_1\cap B_2)$, but $\mu(B_1\cap B_2)=d(B_1\cap B_2,\emptyset)=d(B_1\cap B_2,A_1\cap A_2)<2\varepsilon$. Therefore

$$\left|\mu^*(B_1\cap B_2)-\mu(B_1)-\mu(B_2)\right|<2\varepsilon.$$

Combining everything together we get:

$$\left|\mu^*(A)-\mu^*(A_1)-\mu^*(A_2)\right|<6\varepsilon.$$

Thus $\mu^*$ is additive.

Check the countable additivity for $A=\sqcup_k A_k$. The inequality $\mu^*(A)\leq \sum_k\mu^*(A_k)$ follows from countable sub-additivity. The opposite inequality is the limiting case of the finite inequality $\mu^*(A)\geq$

**Notes**

$\mu^*(\sqcup_{k=1}^{N} A_k)=\sum_{k=1}^{N}\mu^*(A_k)$ following from additivity and monotonicity of $\mu^*$.

**Corollary 14.3.9** *Let $E\subseteq\mathbb{R}$ be open or closed. Then E is Lebesgue measurable.*

**Proof.** This is a common trick, using the density and the countability of the rationals. As σ-algebras are closed under taking complements, we need only show that open sets are Lebesgue measurable.

Intervals $(a,b)$ are Lebesgue measurable by the very definition. Now let $U\subseteq\mathbb{R}$ be open. For each $x\in U$, there exists $a_x<b_x$ with $x\in(a_x,b_x)\subseteq U$. By making $a_x$ slightly larger, and $b_x$ slightly smaller, we can ensure that $a_x,b_x\in\mathbb{Q}$. Thus $U = \cup_x (a_x, b_x)$. Each interval is measurable, and there are at most a countable number of them (endpoints make a countable set) thus $U$ is the countable (or finite) union of Lebesgue measurable sets, and hence $U$ is Lebesgue measurable itself.

We perform now an extension of finite measure to σ-finite one. Let μ be a σ-additive and σ-finite measure defined on a semiring in $X=\sqcup_k X_k$, such that the restriction of μ to every $X_k$ is finite. Consider the Lebesgue extension $\mu_k$ of μ defined within $X_k$. A set $A\subset X$ is measurable if every intersection $A\cap X_k$ is $\mu_k$ measurable. For a such measurable set $A$ we define its measure by the identity:

$$\mu(A)=\sum_k \mu_k(A\cap X_k).$$

We call a measure μ defined on $L$ *complete* if whenever $E\subseteq X$ is such that there exists $F\in L$ with $\mu(F)=0$ and $E\subseteq F$, we have that $E\in L$. Measures constructed from outer measures by the above theorem are always complete. On the example sheet, we saw how to form a complete measure from a given measure. We call sets like *E null sets*: complete measures are useful, because it is helpful to be able to say that null sets are in our σ-algebra. Null sets can be quite complicated. For the Lebesgue measure, all countable subsets of $\mathbb{R}$ are null, but then so is the Cantor set, which is uncountable.

**Definition 14.3.10** If we have a property P(x) which is true except possibly $x \in A$ and $\mu(A)=0$, we say P(x) is almost everywhere or a.e..

**CHECK YOUR PROGRESS**

1. Define countably additive

2. Explain : Any measure $\mu'$ on a semiring S is uniquely extended to a measure $\mu$ on the generated ring R(S). If the initial measure was $\sigma$-additive, then the extension is $\sigma$-additive as well.

3. What is outer measure & explain its properties.

# 14.4 COMPLEX-VALUED MEASURES AND CHARGES

We start from the following observation.

**Definition 14.4.*1*** Let $X$ be a set, and $R$ be a $\sigma$-ring. A real- (complex-) valued function $v$ on $R$ is called a *charge* (or *signed measure*) if it is countably additive as follows: for any $A_k \in R$ the identity $A = \sqcup_k A_k$ implies the series $\sum_k v(A_k)$ is absolute convergent and has the sum $v(A)$.
*In the following "charge" means "real charge".*

**Example** Any linear combination of $\sigma$-additive measures on $\mathbb{R}$ with real (complex) coefficients is real (complex) charge.
The opposite statement is also true:

**Theorem 14.4.2** Any real (complex) charge $\nu$ has a representation $\nu=\mu_1-\mu_2$ ($\nu=\mu_1-\mu_2+i\mu_3-i\mu_4$), where $\mu_k$ are $\sigma$-additive measures.

To prove the theorem we need the following definition.

**Definition 14.4.3** The *variation of a charge* on a set $A$ is $|\nu|(A)=\sup \sum_k |\nu(A_k)|$ for all disjoint splitting $A=\sqcup_k A_k$.

**Example***If* $\nu=\mu_1-\mu_2$, *then* $|\nu|(A)\leq \mu_1(A)+\mu_2(A)$. *The inequality becomes an identity for* disjunctive measures *on A (that is there is a partition* $A=A_1\sqcup A_2$ *such that* $\mu_2(A_1)=\mu_1(A_2)=0$*).*

The relation of variation to charge is as follows:

**Theorem 14.4.4** For any charge $\nu$ the function $|\nu|$ is a $\sigma$-additive measure.*Finally to prove the Thm. 14.3.2 we use the following*

**Proposition 14.4.5** For any charge $\nu$ the function $|\nu|-\nu$ is a $\sigma$-additive measure as well. From the Thm. 14.3.2 we can deduce

**Corollary 14.4.6** The collection of all charges on a $\sigma$-algebra $R$ is a linear space which is complete with respect to the distance:

$$d(\nu_1,\nu_2)=\sup_{A\in R}\left|\nu_1(A)-\nu_2(A)\right|.$$

The following result is also important:

**Theorem 14.4.7 (Hahn Decomposition)** Let $\nu$ be a charge. There exist $A,B\in L$, called a *Hahn decomposition* of $(X,\nu)$, with $A\cap B=\emptyset$, $A\cup B=X$ and such that for any $E\in L$,

$$\nu(A\cap E)\geq 0,\quad \nu(B\cap E)\leq 0.$$

This need not be unique.

**Proof.** We only sketch this. We say that $A \in L$ is *positive* if

$$\nu(E \cap A) \geq 0 \qquad (E \in L),$$

and similiarly define what it means for a measurable set to be *negative*. Suppose that ν never takes the value $-\infty$ (the other case follows by considering the charge $-\nu$). Let $\beta = \inf \nu(B_0)$ where we take the infimum over all negative sets $B_0$. If $\beta = -\infty$ then for each $n$, we can find a negative $B_n$ with $\nu(B_n) \leq -n$. But then $B = \cup_n B_n$ would be negative with $\nu(B) \leq -n$ for any $n$, so that $\nu(B) = -\infty$ a contradiction.

So $\beta > -\infty$ and so for each $n$ we can find a negative $B_n$ $\nu(B_n) < \beta + 1/n$. Then we can show that $B = \cup_n B_n$ is negative, and argue that $\nu(B) \leq \beta$. As $B$ is negative, actually $\nu(B) = \beta$. There then follows a very tedious argument, by contradiction, to show that $A = X \setminus B$ is a positive set. Then $(A,B)$ is the required decomposition.

## 14.5 CONSTRUCTING MEASURES, PRODUCTS

Consider the semiring $S$ of intervals $[a,b)$. There is a simple description of all measures on it. For a measure μ define

$$F_\mu(t) = \begin{cases} \mu([0,t)) & \text{if } t > 0, \\ 0 & \text{if } t = 0, \\ -\mu([t,0)) & \text{if } t < 0, \end{cases} \qquad \text{(A)}$$

$F_\mu$ is monotonic and any monotonic function $F$ defines a measure μ on $S$ by the by $\mu([a,b)) = F(b) - F(a)$. The correspondence is one-to-one with the additional assumption $F(0) = 0$.

**Theorem 14.5.1** The above measure $\mu$ is $\sigma$-additive on $S$ if and only if $F$ is continuous from the left: $F(t-0) = F(t)$ for all $t \in \mathbb{R}$.

Notes

**Proof.** Necessity: $F(t)-F(t-0)=\lim_{\varepsilon\to 0}\mu([t-\varepsilon,t))=\mu(\lim_{\varepsilon\to 0}[t-\varepsilon,t))=\mu()=0$, by the continuity of a σ-additive measure.

For sufficiency assume $[a,b)=\sqcup_k [a_k,b_k)$. The inequality $\mu([a,b))\geq \sum_k \mu([a_k,b_k))$ follows from additivity and monotonicity. For the opposite inequality take $\delta_k$ s.t. $F(b)-F(b-\delta)<\varepsilon$ and $F(a_k)-F(a_k-\delta_k)<\varepsilon/2^k$ (use left continuity of $F$). Then the interval $[a,b-\delta]$ is covered by $(a_k-\delta_k,b_k)$, due to compactness of $[a,b-\delta]$ there is a finite subcovering. Thus

$$\mu([a,b-\delta\ ))\leq\sum_{j=1}^{N} \mu([a_{kj}-\delta_{kj},b_{kj})).$$

**CHECK YOUR PROGRESS**

4. What is charge?

5. Provide statement for Hahn Decomposition

# 14.6 LET'S SUM UP

We Understood the Basic Measure Theory and Extension of Measures

We got clarity about Characters of Commutative Groups and Understood the Complex-Valued Measures and Charges How to Construct Measures, Products

# 14.7 KEYWORDS

**Consistency** - In **mathematics** and in particularly in algebra, a linear or nonlinear system of equations is called as **consistent** if there is at least one set of values for the unknowns that satisfies each equation in the system—that is, that when substituted into each of the equations makes each equation hold true as an identity.

***Semi-ring*** - In abstract algebra, a **semiring** is an algebraic structure similar to a **ring**, but without the requirement that each element must have an additive inverse.

A **monotonic function** is a **function** which is either entirely nonincreasing or nondecreasing. A **function** is **monotonic** if its first derivative (which need not be continuous) does not change sign.

## 14.8 QUESTIONS FOR REVIEW

1. Show that the empty set belongs to any non-empty ring.
2. Let S be a semiring. Show that the collection of all finite disjoint unions $\sqcup_{k=1}^{n} A_k$, where $A_k \in S$, is a ring. We call it the ring R(S) generated by the semiring S.
3. Show that measurability by Lebesgue and Carathéodory are equivalent.

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## 14.10 ANSWER TO CHECK YOUR PROGRESS

1. Provide definition– 14.2.4

2. Provide proof–14.3.1

3. Provide definition–14.3.2 & 14.3.3

4. Provide definition and example – 14.4.1

5. Provide statement – 14.4.7